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Mathematical Problems in Mechanics

The Navier–Stokes equations with Navier's boundary condition around moving bodies in presence of collisions

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Abstract

In this Note, we treat the Navier–Stokes equation with slip Navier's boundary condition in a time variable domain around a finite system of compact bodies moving in a container. The motion of the bodies is assumed to be a priori known. The bodies may collide at a finite number of time instants. We present the theorem on the global in time existence of a weak solution. It is remarkable that Navier's boundary condition enables us to consider a larger class of possible collisions of bodies with C^2 front surfaces in comparison with the no-slip Dirichlet condition. ***To cite this article: J. Neustupa, P. Penel, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Équations de Navier–Stokes avec conditions aux limites de Navier pour décrire un fluide autour de corps en mouvement pouvant entrer en collision. Dans cette Note, nous considérons les équations de Navier–Stokes avec des conditions aux limites de Navier dans un domaine borné temporellement variable contenant un nombre fini de corps compacts en mouvement. Le mouvement de ces corps est supposé connu, ainsi que la simulation de leurs contacts ou collisions éventuels (en nombre fini, entre eux ou avec la frontière du domaine). Nous établissons un résultat d'existence, globale en temps, des solutions faibles. Le choix des conditions aux limites est intéressant à commenter par comparaison avec les conditions standard de Dirichlet. ***Pour citer cet article : J. Neustupa, P. Penel, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Nous nous intéressons aux écoulements visqueux incompressibles, à l'intérieur de D, un domaine borné de $\mathbb{R}^3$, dans la situation assez générale où le fluide entoure K corps solides en mouvement, par conséquent à l'intérieur d'un domaine temporellement variable. Nous notons B_k^t les parties compactes de D que les corps occupent au temps t, $0 \leqslant t < T$, $1 \leqslant k \leqslant K$; pour certaines valeurs de t que nous noterons $t_i^c \in \mathcal{T}^c$, les contacts ou les collisions des B_k^t

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(entre eux ou avec le bord de D) ne sont pas exclus, toutefois nous supposerons connues $\overline{\mathbf{V}}_k(t)$ et $\bar{\boldsymbol{\omega}}_k(t)$ les vitesses de translation et de rotation (centrées en $\bar{\mathbf{x}}_k(t)$) pour chacune des B_k^t, toutes fonctions de classe C^2 en dehors de $\mathcal{T}^c$.

Si $\Omega^t = D \setminus \bigcup_{k=1}^K B_k^t$ et $\Gamma^t = \partial\Omega^t$ avec une régularité suffisante (lipschitzienne, de classe C^1 par morceaux), nous étudions le système des équations de Navier–Stokes dans $\{(\mathbf{x},t) \in \mathbb{R}^4;\ 0 < t < T,\ \mathbf{x} \in \Omega^t\}$ associées aux conditions aux limites suivantes, de type Navier avec un coefficient de frottement γ,

$$\mathbf{v}\cdot\mathbf{n} = \mathbf{V}\cdot\mathbf{n},$$
$$\big[2\nu\,(\nabla\mathbf{v})_s\cdot\mathbf{n}\big]_\tau + \gamma\,(\mathbf{v}-\mathbf{V}) = \mathbf{0},$$

où $\nu > 0$ est le paramètre de viscosité, $\mathbf{n}$ la normale extérieure à Γ^t, où $\mathbf{v}$, $(\nabla\mathbf{v})_s$ et $\mathbf{V}$ désignent successivement la vitesse du fluide, son gradient symétrique, et la vitesse des "points matériels" $\mathbf{x} \in \partial B_k^t$ c'est-à-dire $\mathbf{V}(\mathbf{x},t) := \overline{\mathbf{V}}_k(t) + \bar{\boldsymbol{\omega}}_k(t) \times [\mathbf{x} - \bar{\mathbf{x}}_k(t)]$ pour $t \in (0,T) \setminus \mathcal{T}^c$.

Les étapes cruciales de cette étude sont la construction d'une fonction auxilliaire $\mathbf{a} := \mathbf{a}(\mathbf{x},t)$ de divergence nulle permettant d'assurer l'imperméabilité $(\mathbf{v}-\mathbf{a})\cdot\mathbf{n} = \mathbf{0}$ aux frontières, et la recherche des hypothèses que doit vérifier cette fonction $\mathbf{a}$ pour aboutir à toutes les estimations sans lesquelles une théorie générale d'existence de solutions faibles serait hors d'atteinte : Ces hypothèses dépendent de la géométrie et du mouvement des B_k^t, elles sont décrites dans la partie anglaise, nous les avons soigneusement vérifiées dans le cas de deux corps de formes quasi-hémisphériques aux voisinages des points de contact ou de collision. Les majorations exigent également beaucoup de soin pour obtenir des constantes indépendantes de Ω^t.

Le résultat principal est donc un théorème d'existence de solutions faibles pour le problème (1)–(5), outre les hypothèses déjà mentionnées, les vitesses de possibles collisions aux temps t_i^c doivent rester petites. S'affranchir de cette contrainte semble difficile. Avec cette contrainte, l'importance du choix des conditions aux limites est ici remarquable, car l'existence de solutions faibles pour le système des équations de Navier–Stokes associées à des conditions aux limites de Dirichlet n'autorise que des vitesses nulles de possibles collisions aux temps t_i^c, condition sine qua non.

La démonstration est fondée sur une méthode d'approximation par semi-discrétisation temporelle, une étape non évidente étant le passage à la limite dans le terme non linéaire : Nous nous inspirons du travail de K.H. Hoffmann et V.N. Starovoitov dans le cas bidimensionnel, et nous justifions la convergence forte pour des projections locales de Leray–Helmholtz de nos approximations, car il n'est pas possible d'appliquer comme d'habitude le théorème de Aubin–Lions.

1. Introduction

A global (in time) weak solvability of the Navier–Stokes equations with the no-slip Dirichlet boundary condition in a fixed domain $\Omega \subset \mathbb{R}^3$ is a classical result of the qualitative theory of the Navier–Stokes equations. The same result in a time variable domain Ω^t with a prescribed form at each time t was proved by H. Fujita and N. Sauer [1] and it was recently generalized by J. Neustupa [5]. In [1], the boundary of Ω^t consisted of a finite number of moving simple closed surfaces of the class C^3 so that the distance of any two of these surfaces was never less than $d > 0$. In [5], Ω^t has an arbitrary shape and smoothness, the assumptions on Ω^t involve simulation of collisions of bodies moving in a fluid. The existence and uniqueness of a strong solution in domain Ω^t with given smooth moving boundaries was proved by O.A. Ladyzhenskaya [4] (globally in time for sufficiently small data or locally in time for large data).

During the last decade, a series of other works dealing with the motion of bodies in a fluid considers the system fluid–bodies to be interconnected so that the position of the bodies in the fluid is not known in advance. Of all authors who have contributed by papers belonging to this category, let us name e.g. K.H. Hoffmann, V.N. Starovoitov, B. Desjardins, M.J. Esteban, C. Conca, J. San Martín, M. Tucsnak, M.D. Gunzburger, H.C. Lee, G. Seregin, E. Feireisl, and T. Takahashi. All the cited authors consider the homogeneous Dirichlet boundary condition for velocity on the boundary. Other works study the motion of the system fluid–body under the assumption that the body produces a certain velocity profile on its surface and it moves do to this velocity. The survey of results on these so called "self-propelled bodies" was presented by G.P. Galdi [2].

V.N. Starovoitov [7] derived necessary conditions for the existence of a weak solution of the Navier–Stokes equations in a time variable domain Ω^t, exterior to several solid bodies moving in the fluid. The conditions show that if the bodies have boundaries of the class C^2 and the fluid satisfies no-slip Dirichlet's boundary condition then the bodies can strike only with the speed equal to zero at the instant of the collision, otherwise the weak solution cannot exist.

Motivated by this state, we study the solvability of the Navier–Stokes equations in a time-variable domain Ω^t, which is an exterior of several bodies moving in a container, under the assumption that the velocity of the fluid satisfies Navier's slip boundary condition on the boundary Γ^t. We assume that the motion of the bodies is a priori known. We show that the weak solution can exist even if the bodies strike with a non-zero speed.

The "classical" formulation of the studied initial-boundary value problem is

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = \nu\Delta\mathbf{v} + \mathbf{f} \quad \text{a.e. in } Q_{(0,T)}, \tag{1}$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{a.e. in } Q_{(0,T)}, \tag{2}$$

$$\mathbf{v}\cdot\mathbf{n} = \mathbf{V}\cdot\mathbf{n} \quad \text{a.e. in } \Gamma_{(0,T)}, \tag{3}$$

$$\left[\mathbb{T}_{\mathrm{d}}(\mathbf{v})\cdot\mathbf{n}\right]_\tau + \gamma(\mathbf{v}-\mathbf{V}) = \mathbf{0} \quad \text{a.e. in } \Gamma_{(0,T)}, \tag{4}$$

$$\mathbf{v} = \mathbf{v}_0 \quad \text{a.e. in } \Omega^0\times\{0\}, \tag{5}$$

where $Q_{(0,T)} := \{(\mathbf{x},t)\in\mathbb{R}^4;\ 0<t<T,\ \mathbf{x}\in\Omega^t\}$ and $\Gamma_{(0,T)} := \{(\mathbf{x},t)\in\mathbb{R}^4;\ 0<t<T,\ \mathbf{x}\in\Gamma^t\}$.

The unknowns are $\mathbf{v}$ and p. The symbols ν, $\mathbf{f}$, $\mathbf{n}$, $\mathbf{V}$, $\mathbb{T}_{\mathrm{d}}(\mathbf{v})$ and γ successively denote the kinematic coefficient of viscosity, the specific external body force, the outer normal vector on the boundary, the velocity of "material points" on the boundary Γ^t of Ω^t, the dynamic stress tensor associated with the flow $\mathbf{v}$ and γ is the coefficient of friction between the fluid and the boundary. Tensor $\mathbb{T}_{\mathrm{d}}(\mathbf{v})$ has the form $\mathbb{T}_{\mathrm{d}}(\mathbf{v}) = 2\nu(\nabla\mathbf{v})_s$ where $(\nabla\mathbf{v})_s$ is the symmetrized gradient of $\mathbf{v}$. The subscript τ denotes the tangential component to Γ^t.

2. Assumptions on domain Ω^t and notation of norms and function spaces

Let $T>0$. We suppose that K solid bodies move in the fluid in a fixed container D in the time interval $[0,T]$ so that their positions are known in advance. Thus, the time variable domain Ω^t, filled by the fluid, has the form $\Omega^t = D \setminus \bigcup_{k=1}^K B_k^t$ for $0\leqslant t\leqslant T$, where $B_1^t,\dots,B_K^t$ are compact regions occupied by the bodies at time t. The bodies can strike themselves or the boundary of the container at certain critical instants of time $t_1^c,\dots,t_M^c$ in the interval $(0,T)$. We denote the set of these critical times by $\mathcal{T}^c$. We assume that

(i) D and the interiors of sets B_k^t ($k=1,\dots,K$) are Lipschitz domains in $\mathbb{R}^3$ with piecewise C^1 boundaries,

(ii) the translational velocity $\overline{\mathbf{V}}_k(t)$, the rotational velocity $\bar{\boldsymbol{\omega}}_k(t)$ and the center of rotation $\bar{\mathbf{x}}_k(t)$ of each body B_k^t are functions from $C^2([0,T]\setminus\mathcal{T}^c)^3$.

Thus, material points $\mathbf{x}\in B_k^t$ move with the known velocity $\mathbf{V}(\mathbf{x},t) := \overline{\mathbf{V}}_k(t) + \bar{\boldsymbol{\omega}}_k(t)\times[\mathbf{x}-\bar{\mathbf{x}}_k(t)]$ for $t\in(0,T)\setminus\mathcal{T}^c$.

We denote by $L^q_\sigma(\Omega^t)$ (for $1<q<+\infty$) the space of the divergence-free (in the sense of distributions) vector functions from $L^q(\Omega^t)^3$ that have the normal component on the boundary equal to zero (in the sense of traces). The norm in $L^q_\sigma(\Omega^t)$ is denoted by $\|\,.\,\|_{q;\Omega^t}$. Furthermore, we define $W^{1,2}_\sigma(\Omega^t) := W^{1,2}(\Omega^t)^3\cap L^2_\sigma(\Omega^t)$ and we denote the norm in $W^{1,2}_\sigma(\Omega^t)$ by $\|\,.\,\|_{1,2;\Omega^t}$.

The next condition we impose on domain Ω^t is

$$\text{(iii)}\quad \exists c_1>0 \quad \forall t\in[0,T]\setminus\mathcal{T}^c \quad \forall\boldsymbol{\phi}\in W^{1,2}_\sigma(\Omega^t):\ \int_{\Gamma^t}\boldsymbol{\phi}\cdot\nabla\mathbf{n}\cdot\boldsymbol{\phi}\,\mathrm{d}S \leqslant c_1\|\boldsymbol{\phi}\|_{2;\Omega^t}\|\boldsymbol{\phi}\|_{1,2;\Omega^t}.$$

This inequality is not surprising at the first sight: the integral on the left hand side can be naturally estimated as we need by means of an appropriate theorem on traces. However, the problem is that the constant in the inequality we obtain from the theorem on traces generally depends on t. Assumption (iii) thus expresses the requirement that the inequality is satisfied with constant c_1 independent of t.

In order to transform the inhomogeneous boundary condition (3) to the homogeneous one, we look for the solution $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{a}+\mathbf{u}$ where $\mathbf{u}$ is the new unknown function and $\mathbf{a}$ is supposed to be a known divergence-free vector-function, defined in the set $Q_{[0,T]\setminus\mathcal{T}^c}$ such that it takes on the inhomogeneous part of condition (3), i.e. it satisfies $\mathbf{a}\cdot\mathbf{n} = \mathbf{V}\cdot\mathbf{n}$ in $\Gamma_{[0,T]\setminus\mathcal{T}^c}$. Thus, function $\mathbf{u}$ should now satisfy the homogeneous boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ a.e. in $\Gamma_{(0,T)}$.

In order to establish a general theorem on the existence of a weak solution, we need function **a** to satisfy further conditions (a1)–(a5) listed below. This means that the application of the theorem to a particular geometrical configuration always requires to show that an appropriate function **a** satisfying all the named conditions can be constructed. We will sketch such a construction in one concrete situation as an example in Section 4. The mentioned conditions are:

(a1) $\mathbf{a}$ and $\partial_t \mathbf{a}$ are continuous in $Q_{[0,T]} \setminus \mathcal{T}^c$,
(a2) $\theta_1(t) := \|\mathbf{a}(.,t)\|_{1,2;\Omega^t} \in L^2(0,T)$,
(a3) $\theta_2(t) := \|\mathbf{a}(.,t) - \mathbf{V}(.,t)\|_{2;\Gamma^t} \in L^2(0,T)$,
(a4) there exist functions $\theta_3 \in L^1(0,T)$, $\theta_4 \in L^2(0,T)$ and $\theta_5 \in L^1(0,T)$, continuous in $[0,T] \setminus \mathcal{T}^c$, such that for $t \in [0,T] \setminus \mathcal{T}^c$ and $\boldsymbol{\phi} \in W^{1,2}_\sigma(\Omega^t)$ we have

$$\left| \int_{\Omega^t} \left[\partial_t \mathbf{a}(.,t) + \mathbf{a}(.,t) \cdot \nabla \mathbf{a}(.,t)\right] \cdot \boldsymbol{\phi} \, \mathrm{d}\mathbf{x} \right| \leqslant \theta_3(t) \|\boldsymbol{\phi}\|_{2;\Omega^t} + \theta_4(t) \|\nabla \boldsymbol{\phi}\|_{2;\Omega^t}, \tag{6}$$

$$\left| \int_{\Omega^t} \boldsymbol{\phi} \cdot \nabla \boldsymbol{\phi} \cdot \mathbf{a}(.,t) \, \mathrm{d}\mathbf{x} \right| \leqslant \frac{\nu}{10} \|\nabla \boldsymbol{\phi}\|^2_{2;\Omega^t} + \frac{\gamma}{4} \|\boldsymbol{\phi}\|^2_{2;\Gamma^t} + \theta_5(t) \|\boldsymbol{\phi}\|^2_{2;\Omega^t}, \tag{7}$$

(a5) the initial-value problem $\{(\mathrm{d}/\mathrm{d}t)\mathbf{X}(t;\vartheta,\mathbf{x}) = \mathbf{a}(\mathbf{X}(t;\vartheta,\mathbf{x}),t)$ and $\mathbf{X}(\vartheta;\vartheta,\mathbf{x}) = \mathbf{x}\}$ has a unique solution $\mathbf{X}(t;\vartheta,\mathbf{x})$, defined for $t \in [0,T]$, $\vartheta \in [0,T]$ and a.a. $\mathbf{x} \in \Omega^\vartheta$, such that the mapping $\mathbf{x} \mapsto \mathbf{X}(t;\vartheta,\mathbf{x})$ is a one-to-one transformation of $\Omega^\vartheta \setminus \mathfrak{s}^\vartheta$ onto $\Omega^t \setminus \mathfrak{s}^t$ (where $\mathfrak{s}^\vartheta$ and $\mathfrak{s}^t$ are sets of measure zero in Ω^ϑ or in Ω^t, respectively).

The Jacobian of the mapping $\mathbf{x} \mapsto \mathbf{X}(t;\vartheta,\mathbf{x})$ equals one due to the incompressibility of flow **a**.

3. The weak formulation of the problem (1)–(5) and the main theorem

Denote by $\boldsymbol{\phi}$ an infinitely differentiable divergence-free vector-function in $\overline{Q}_{[0,T]}$ that has a compact support in $\overline{Q}_{[0,T)}$ and satisfies the condition $\boldsymbol{\phi} \cdot \mathbf{n} = 0$ a.e. on $\Gamma_{(0,T)}$. Assume that $\mathbf{v}$ is a "sufficiently smooth" solution of (1)–(5) of the form $\mathbf{v} = \mathbf{a} + \mathbf{u}$ where $\mathbf{a}$ satisfies all the assumptions named in Section 2 and $\mathbf{u} \in W^{1,2}_\sigma(\Omega^t)$ for a.a. $t \in (0,T)$. Let us multiply Eq. (1) by function $\boldsymbol{\phi}$ and integrate on $Q_{(0,T)}$. The integral of $\{\partial_t \mathbf{u} + (\mathbf{a} \cdot \nabla)\mathbf{u}\} \cdot \boldsymbol{\phi}$ in Ω^t can be transformed by means of the substitution $\mathbf{x} = \mathbf{X}(t;0,\mathbf{x}_0)$ to the integral on Ω^0. Afterwards, integrating by parts with respect to t and using the backward substitution $\mathbf{x}_0 = \mathbf{X}(0;t,\mathbf{x})$, we get

$$\int_0^T \int_{\Omega^t} \left\{\partial_t \mathbf{u}(\mathbf{x},t) + \mathbf{a}(\mathbf{x},t) \cdot \nabla \mathbf{u}(\mathbf{x},t)\right\} \cdot \boldsymbol{\phi}(\mathbf{x},t) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t$$
$$= -\int_{\Omega^0} \mathbf{u}_0(\mathbf{x}_0) \cdot \boldsymbol{\phi}(\mathbf{x}_0,0) \, \mathrm{d}\mathbf{x}_0 - \int_0^T \int_{\Omega^t} \mathbf{u}(\mathbf{x},t) \cdot \left\{\partial_t \boldsymbol{\phi}(\mathbf{x},t) + \mathbf{a}(\mathbf{x},t) \cdot \nabla \boldsymbol{\phi}(\mathbf{x},t)\right\} \mathrm{d}\mathbf{x} \, \mathrm{d}t,$$

where $\mathbf{u}_0 = \mathbf{v}_0 - \mathbf{a}(.,0)$. The other integral which must be treated in a non-standard way is the integral of $\nu \Delta \mathbf{v} \cdot \boldsymbol{\phi}$: applying the integration by parts and using essentially the boundary condition (4), we obtain

$$\int_{\Omega^t} \nu \Delta \mathbf{v} \cdot \boldsymbol{\phi} \, \mathrm{d}\mathbf{x} = -\int_{\Gamma^t} \gamma (\mathbf{v} - \mathbf{V}) \cdot \boldsymbol{\phi} \, \mathrm{d}S - \int_{\Omega^t} 2\nu (\nabla \mathbf{v})_s : \nabla \boldsymbol{\phi} \, \mathrm{d}\mathbf{x}. \tag{8}$$

Writing everywhere $\mathbf{a} + \mathbf{u}$ instead of $\mathbf{v}$, we finally obtain the integral equation

$$\int_0^T \int_{\Omega^t} \left\{-(\partial_t \boldsymbol{\phi} + \mathbf{a} \cdot \nabla \boldsymbol{\phi}) \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \boldsymbol{\phi} \cdot \mathbf{a} + \mathbf{u} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\phi} + 2\nu \left[\nabla(\mathbf{a} + \mathbf{u})\right]_s : \nabla \boldsymbol{\phi}\right\} \mathrm{d}\mathbf{x} \, \mathrm{d}t$$
$$+ \int_0^T \int_{\Gamma^t} \gamma (\mathbf{a} + \mathbf{u} - \mathbf{V}) \cdot \boldsymbol{\phi} \, \mathrm{d}S \, \mathrm{d}t = \int_0^T \int_{\Omega^t} \mathbf{g} \cdot \boldsymbol{\phi} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_{\Omega^0} \mathbf{u}_0 \cdot \boldsymbol{\phi}(.,0) \, \mathrm{d}\mathbf{x} \tag{9}$$

where $\mathbf{g} = \mathbf{f} - \partial_t \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{a}$. We arrive at the definition:

Definition 3.1. Suppose that $\mathbf{u}_0 \in L^2_\sigma(\Omega^0)$ and $\mathbf{f} \in L^2(0, T; L^2(\Omega^t)^3)$. We call the function $\mathbf{v} \equiv \mathbf{a} + \mathbf{u}$ a *weak solution* of the problem (1)–(5) if

- $\mathbf{u} \in L^2(0, T; W^{1,2}_\sigma(\Omega^t)) \cap L^\infty(0, T; L^2_\sigma(\Omega^t))$,
- the trace of $\mathbf{u}$ on $\Gamma_{(0,T)}$ is in $L^2(0, T; L^2(\Gamma^t)^3)$ and
- $\mathbf{u}$ satisfies (9) for all test functions $\boldsymbol{\phi}$ with the named properties.

Our main theorem reads:

Theorem 3.1. *Suppose that domain Ω^t satisfies all the conditions* (i)–(iii). *Suppose that there exists a divergence-free function* $\mathbf{a}$ *in $Q_{[0,T] \smallsetminus \mathcal{T}^c}$, such that* $\mathbf{a} \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n}$ *in $\Gamma_{[0,T] \smallsetminus \mathcal{T}^c}$ and* $\mathbf{a}$ *satisfies conditions* (a1)–(a5) *from Section* 2. *Then there exists a weak solution of the problem* (1)–(5).

The proof of this theorem is quite lengthy and technical and it is based on the construction of Rothe approximations. The details can be found in [6]. The main tool, which we apply in order to deduce that the sequence of the approximations contains a sub-sequence that converges weakly (respectively weakly – $*$) in certain function spaces, are energy-type inequalities for the approximations. In order to obtain these inequalities, we need condition (iii) and assumptions (a1)–(a5).

As usually, the finest part of the proof concerns the limit transition in the nonlinear term in (9). In order to do it, we need a piece of information on a strong convergence of the approximations. Here, due to the variability of domain Ω^t, we cannot apply the Aubin–Lions lemma in a standard way. We prove a kind of "interior strong convergence" of certain local Leray–Helmholtz projections of the approximations, which turns out to be sufficient for passing to the limit. A similar idea was already used by K.H. Hoffmann, V.N. Starovoitov in [3].

4. Example: The flow around two striking bodies with ball-shaped front surfaces

In this section, we assume that two compact bodies B^t_1 and B^t_2 move in $\mathbb{R}^3$ in the time interval $[0, T]$ and they strike at the time instant $t^c \in (0, T)$. Thus, the time-variable domain Ω^t has the form $\Omega^t = \mathbb{R}^3 \smallsetminus (B^t_1 \cup B^t_2)$ and set $\mathcal{T}^c$ of critical times in $(0, T)$ is the one point set $\mathcal{T}^c = \{t^c\}$. We assume that conditions (i) and (ii) from Section 2 are fulfilled (with $D = \mathbb{R}^3$ and $K = 2$). Furthermore, we assume that

(iv) bodies B^t_1 and B^t_2 touch themselves at time t^c by material points $P^t_1 \in \partial B^t_1$ and $P^t_2 \in \partial B^t_2$, in whose neighbourhoods the surfaces of B^t_1 and B^t_2 coincide with surfaces S^t_1 and S^t_2 of the balls with the radii R_1 and R_2.

Conditions (i), (ii) and (iv) imply that there exists $\tau > 0$ such that for t in the time interval $(t^c - \tau, t^c + \tau)$, the shortest line segment ℓ^t connecting B^t_1 and B^t_2 has the end points on surfaces S^t_1 and S^t_2. The length δ^t of ℓ^t, as a function of variable t, is continuous on $[0, T]$ and such that $\delta^t = 0$ for $t = t^c$ and $\delta^t > 0$ for $t \in [0, t^c) \cup (t^c, T]$. Moreover, it belongs to $C^2([0, t^c) \cup (t^c, T])$. Furthermore, conditions (i), (ii) and (iv) imply (iii).

In order to apply Theorem 3.1 in this geometrical configuration, we need to construct function $\mathbf{a}$, satisfying the equation of continuity (2), the boundary condition (3) and assumptions (a1)–(a5). It is advantageous to define function $\mathbf{a}$ in a Cartesian coordinate system y^t_1, y^t_2, y^t_3 such that ℓ^t is a subset of the y^t_3-axis and the origin O^t is in the middle of ℓ^t. This system can be chosen so that the linear transformation between the Cartesian coordinates x_1, x_2, x_3 and y^t_1, y^t_2, y^t_3 is smooth, i.e. its coefficients are functions from $C^2([0, t^c) \cup (t^c, T])$, continuous on $[0, T]$. Function $\mathbf{a}$ can be at first defined in a "critical sub-domain" Ω^t_c of Ω^t which contains the line segment ℓ^t and it coincides with Ω^t in the neighbourhood of the point of the collision of bodies B^t_1 and B^t_2, at times close to the critical time t^c. Afterwards, $\mathbf{a}$ can be appropriately extended to the whole domain Ω^t and for other times $t \in [0, t^c) \cup (t^c, T]$. The construction of $\mathbf{a}$ in domain Ω^t_c is based on the definition of a vectorial potential $\mathbf{w}$ in the region between B^t_1 and B^t_2. Function $\mathbf{a}$ is defined to be equal to $\dot{\delta}^t\, \mathbf{curl}\, \mathbf{w}$ in Ω^t_c. The quantity $\dot{\delta}^t$ expresses the relative speed of the bodies B^t_1 and B^t_2. This definition of $\mathbf{a}$ guarantees that $\mathbf{a}$ is divergence-free. The validity of (3) as well as conditions (a1)–(a5) crucially depends on the form and properties of the function $\mathbf{w}$. The details can be found in our paper [6]. It is important to mention that the validity of (a4), namely inequality (7), leads to the restriction, that $|\dot{\delta}^t|$ is "sufficiently small" in comparison with coefficients ν and γ for t in a certain neighbourhood of the instant of collision t^c. Then we can apply Theorem 3.1 and obtain the statement on the global in time existence of a weak solution to the problem (1)–(5).

In addition to the considered situation, when bodies B_1^t and B_2^t strike with ball-like surfaces, we also discuss the case of more general front surfaces of B_1^t and B_2^t in paper [6]. We give a hint how to construct the auxiliary function **a**. The verification of its required properties (a1)–(a5), as well as other details, however, will be the contents of another prepared paper.

Remark 4.1. We recall that the same result is impossible if Dirichlet's no-slip boundary condition is considered instead of Navier's conditions (3), (4). (In that case, the speed $|\dot{\delta}^t|$ must tend to zero as $t \to t^c$; otherwise the weak solution does not exist.)

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