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A remark on a Note by Laguerre

Jacques Peyrière [a,b]

[a] *Université Paris-Sud, Mathématique bâtiment 425, CNRS UMR 8628, 91405 Orsay cedex, France*
[b] *Department of Mathematics, Tsinghua University, Beijing, 100084, PR China* [1]

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Abstract

An improvement on Laguerre's method of location of zeroes of orthogonal polynomials is given. ***To cite this article: J. Peyrière, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sur une Note de Laguerre. On donne une amélioration de la méthode de Laguerre de localisation des zéros des polynômes orthogonaux. ***Pour citer cet article : J. Peyrière, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Dans sa Note [1], Laguerre donne un procédé pour majorer la plus grande racine d'un polynôme dont tous les zéros sont réels. Il l'applique aux polynômes orthogonaux. Sa méthode repose sur l'assertion suivante où f est un polynôme de degré $n-1$: « Il est clair que l'équation $f(x)=0$ a également toutes ses racines réelles ; par suite son hessien $(n-2)f'^2(x)-(n-1)f(x)f''(x)$ ne peut avoir une valeur négative ».

Nous donnons une forme plus générale de cette dernière assertion, ce qui permet d'améliorer la borne donnée par Laguerre.

Nous dirons qu'une fraction rationnelle $R \in \mathbb{R}(\sigma_1, \ldots, \sigma_m)$ est *conditionnellement positive*, et nous écrirons alors $R \in \mathrm{CP}_m$, si elle est positive ou nulle, lorsqu'elle est finie, quand les σ_j sont les fonctions symétriques élémentaires de m nombres réels. Si elle est positive lorsque les σ_j sont les fonctions symétriques de m nombres positifs, nous écrirons $R \in \mathrm{CP}_m^+$. Ainsi, nous avons par exemple

$$\sigma_1^2 - 4\sigma_2 \in \mathrm{CP}_2 \quad \text{et} \quad \sigma_1\sigma_2 - 9\sigma_3 \in \mathrm{CP}_3^+.$$

La proposition suivante permet de construire des éléments de CP_m et de CP_m^+.

E-mail address: Jacques.Peyriere@math.u-psud.fr.
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Proposition 0.1. *Si $f(\sigma_1,\dots,\sigma_m)\in \mathrm{CP}_m$, alors, pour $n\geqslant m$, on a*

$$f\left(\binom{m}{1}\binom{n}{1}^{-1}\sigma_1,\binom{m}{2}\binom{n}{2}^{-1}\sigma_2,\dots,\binom{m}{m}\binom{n}{m}^{-1}\sigma_m\right)\in \mathrm{CP}_n.$$

On a le même énoncé pour CP^+ *au lieu de* CP.

En appliquant cette proposition à $\sigma_1^2-4\sigma_2$, on obtient

$$(n-1)\sigma_1^2-2n\sigma_2\in \mathrm{CP}_n\quad \text{pour } n\geqslant 2.$$

Proposition 0.2. *Soit $f\in \mathrm{CP}_n$. Si le polynôme $P\in\mathbb{R}[x]$ de degré au plus n a toutes ses racines réelles, alors la fraction*

$$\varphi(x)=f\left(\frac{P'(x)}{1!P(x)},\frac{P''(x)}{2!P(x)},\dots,\frac{P^{(n)}(x)}{n!P(x)}\right)$$

ne prend que des valeurs positives. Si f appartient seulement à CP_n^+, alors $\varphi(x)\geqslant 0$ pourvu que x soit plus grand que la plus grande racine de P.

Continuant l'exemple précédent, on obtient $(n-1)P'^2-nPP''\geqslant 0$ si P est un polynôme de degré au plus n dont toutes les racines sont réelles. C'est l'assertion utilisée par Laguerre. En partant de $\sigma_1\sigma_2-9\sigma_3\in \mathrm{CP}_3^+$, on obtient $(n-2)\sigma_1\sigma_2-3n\sigma_3\in \mathrm{CP}_n^+$ et $(n-2)P'(x)P''(x)-nP(x)P'''(x)\geqslant 0$ si x est supérieur à la plus grande des racines de P, toutes supposées réelles.

Proposition 0.3. *Soit f dans CP_{n-1} et P un polynôme dans $\mathbb{R}[x]$ de degré au plus n dont toutes les racines sont réelles. Alors, pour toute racine simple a de P, on a*

$$f\left(\frac{P''(a)}{2!P'(a)},\frac{P^{(3)}(a)}{3!P'(a)},\dots,\frac{P^{(n)}(a)}{n!P'(a)}\right)\geqslant 0.$$

Si f est seulement dans CP_{n-1}^+, on peut seulement affirmer que l'inégalité précédente a lieu lorsque a est la plus grande racine de P.

Dans le cas où P est un polynôme orthogonal, les quantités $P^{(j)}(a)/P'(a)$ se calculent par récurrence en utilisant l'équation différentielle satisfaite par P.

Partant de $\sigma_1^2-4\sigma_2\in \mathrm{CP}_2$, on obtient, comme Laguerre, que les racines de H_n, le $n^{\text{ième}}$ polynôme de Hermite, sont inférieures à $\frac{2(n-1)}{\sqrt{n+2}}$. En partant de $\sigma_1\sigma_2-9\sigma_3\in \mathrm{CP}_3^+$, on obtient la borne $\sqrt{\frac{(4n+3)(n-1)}{n+3}}$.

1. Introduction

In his Note [1], Laguerre gives a way of getting an upper bound for the largest root of a polynomial all the zeroes of which are real. This relies on the following assertion "*The polynomial f of degree n has all its roots real, therefore its Hessian $(n-1)f'^2(x)-nf(x)f''(x)$ cannot assume negative values…*". This note provides a more general result.

Just as Laguerre did, we apply our method to get improved bounds for the largest root of orthogonal polynomials.

2. Conditionally positive rational fractions

If $x_1,x_2,\dots,x_n$ are numbers or indeterminates one considers their elementary symmetric functions $\{\sigma_j\}_{1\leqslant j\leqslant n}$ defined by the formula

$$x^n+\sum_{j=1}^{n}(-1)^j\sigma_jx^{n-j}=\prod_{j=1}^{n}(x-x_j).$$

One also sets $\sigma_0=1$ and $\sigma_k=0$ for $k>n$.

Definition 2.1 *(Conditional positivity).* A rational fraction $f(\sigma_1, \dots, \sigma_m) \in \mathbb{R}(\sigma_1, \dots, \sigma_m)$ is said to be in CP_m (respectively in CP_m^+) if $f(\sigma_1, \dots, \sigma_m) \geqslant 0$ each time $\sigma_1, \dots, \sigma_m$ are the elementary symmetric functions of k real numbers, with $k \leqslant m$ (respectively of k nonnegative numbers) and $|f(\sigma_1, \dots, \sigma_m)|$ is finite.

Examples 1.

(i) $\sigma_1^2 - 4\sigma_2 \in \mathrm{CP}_2$.
(ii) $\sigma_1^2 - 2\sigma_2 \in \bigcap_{m \geqslant 2} \mathrm{CP}_m$.
(iii) $\sigma_1\sigma_2 - 9\sigma_3 \in \mathrm{CP}_3^+$ and $\sigma_1\sigma_3 - 16\sigma_4 \in \mathrm{CP}_4^+$.
(iv) The discriminant of $x^n + \sum_{j=1}^n (-1)^j \sigma_j x^{n-j}$ is in CP_n.
(v) $\sigma_1^m - m^m \sigma_m \in \mathrm{CP}_m^+$ for any $m \geqslant 2$.
(vi) Let k and n be two integers larger than 1, and $l > 0$. For positive numbers $x_1, \dots, x_n$, one has

$$\left(x_1^l + \dots + x_n^l\right)^k \leqslant n^{k-1}\left(x_1^{kl} + \dots + x_n^{kl}\right).$$

So, by expressing the Newton sums in terms of the elementary symmetric polynomials, one gets an element of CP_n^+ (of CP_n if kl is even).
More generally, since the moments are log-convex, one has

$$\left(x_1^q + \dots + x_n^q\right)^{r-p} \leqslant \left(x_1^p + \dots + x_n^p\right)^{r-q}\left(x_1^r + \dots + x_n^r\right)^{q-p},$$

if the integers n, p, q, and r fulfill the conditions $n \geqslant 2$ and $1 \leqslant p < q < r$. Again, by expressing the Newton sums in terms of the elementary symmetric polynomials, one gets an element of CP_n^+ (of CP_n if both p and r are even).

The following two propositions give ways of getting new elements in CP_n or CP_n^+.

Proposition 2.2. *If $f(\sigma_1, \dots, \sigma_m) \in \mathrm{CP}_m$, then, for $n \geqslant m$,*

$$f\left(\binom{m}{1}\binom{n}{1}^{-1}\sigma_1, \binom{m}{2}\binom{n}{2}^{-1}\sigma_2, \dots, \binom{m}{m}\binom{n}{m}^{-1}\sigma_m\right) \in \mathrm{CP}_n.$$

The same holds for CP^+ *instead of* CP.

Proof. Indeed, if the roots of the polynomial $P = x^n + \sum_{j=1}^n (-1)^j \sigma_j x^{n-j}$ are real (respectively nonnegative) so are the ones of the $(n-m)$-th derivative of P. □

Example 1. By applying this proposition with $m = 2$ and $f(\sigma_1, \sigma_2) = \sigma_1^2 - 4\sigma_2$, one gets $(n-1)\sigma_1^2 - 2n\sigma_2 \in \mathrm{CP}_n$, for any $n \geqslant 2$. As a matter of fact, this is also a particular case of Example 1.vi.

Proposition 2.3. *Let f be a polynomial in $\mathbb{R}[\sigma_1, \dots, \sigma_n]$ of degree d. Consider the polynomial*

$$g(\sigma_1, \dots, \sigma_n) = \sigma_n^d f\left(\frac{\sigma_{n-1}}{\sigma_n}, \frac{\sigma_{n-2}}{\sigma_n}, \dots, \frac{\sigma_1}{\sigma_n}, \frac{1}{\sigma_n}\right).$$

Then

(i) *If $f \in \mathrm{CP}_n^+$, then $g \in \mathrm{CP}_n^+$,*
(ii) *if d is even, $f \in \mathrm{CP}_n$ implies $g \in \mathrm{CP}_n$.*

Proof. Indeed, if $\sigma_n \neq 0$, the roots of $x^n + \sum_{j=1}^{n-1} (-1)^j \frac{\sigma_{n-j}}{\sigma_n} x^{n-j} + (-1)^n \frac{1}{\sigma_n}$ are the inverses of those of $x^n + \sum_{j=1}^n (-1)^j \sigma_j x^j$. □

3. Polynomials whose all roots are real

Proposition 3.1. *Let $f \in \mathrm{CP}_n$. If the polynomial $P \in \mathbb{R}[x]$ of degree at most n has all its roots real, then the fraction*

$$\varphi(x) = f\left(\frac{P'(x)}{1!P(x)}, \frac{P''(x)}{2!P(x)}, \ldots, \frac{P^{(n)}(x)}{n!P(x)}\right)$$

assume non-negative values only. If f merely is in CP_n^+, then $\varphi(x) \geqslant 0$ provided that x be larger than the largest root of P.

Proof. Set $P(x) = \alpha \prod_{j=1}^n (x - x_j)$. Let $\sigma_1, \ldots, \sigma_n$ stand for the elementary symmetric functions of the variables $\{\frac{1}{x-x_j}\}_{1\leqslant j\leqslant n}$. Then, for $1 \leqslant k \leqslant n$, one has

$$\frac{P^{(k)}(x)}{P(x)} = k!\sigma_k,$$

from which the result follows. □

Example 2. We know (Example 1) that $(n-1)\sigma_1^2 - 2n\sigma_2 \in \mathrm{CP}_n$. So, if P is a polynomial of degree at most n whose roots are real, then $(n-1)P'^2 - nPP'' \geqslant 0$. This is the statement in Laguerre's note which motivated the present work.

Proposition 3.2. *Let f be in CP_{n-1} and P a polynomial in $\mathbb{R}[x]$ of degree at most n whose roots are real. Then, for any simple root a of P, one has*

$$f\left(\frac{P''(a)}{2!P'(a)}, \frac{P^{(3)}(a)}{3!P'(a)}, \ldots, \frac{P^{(n)}(a)}{n!P'(a)}\right) \geqslant 0.$$

If f merely is in CP_{n-1}^+, one only can assert that the previous inequality holds when a is the largest root of P.

Proof. Apply the preceding proposition to the polynomial $P(x)/(x-a)$. □

4. Location of zeroes of orthogonal polynomials

Suppose that a polynomial P of degree n whose all roots are real satisfies a differential equation of the form

$$P''(x) - u(x)P'(x) - v(x)P(x) = 0,$$

where u and v are polynomials.

One has

$$\begin{aligned} P^{(3)} &= uP'' + (u'+v)P' + v'P = u(uP'+vP) + (u'+v)P' + v'P \\ &= \big(u^2 + u' + v\big)P' + (uv + v')P, \end{aligned}$$

and

$$\begin{aligned} P^{(4)} &= \big(u^2 + u' + v\big)P'' + (2uu' + u'' + uv + 2v')P' + (uv' + u'v + v')P \\ &= \big(u^2 + u' + v\big)(uP' + vP) + (2uu' + u'' + uv + 2v')P' + (uv' + u'v + v')P \\ &= \big(u^3 + 3uu' + 2uv + u'' + 2v'\big)P' + \big(u^2v + 2u'v + v^2 + uv' + v'\big)P, \end{aligned}$$

and so on.

So, if a is a root of P, for any $k \geqslant 2$, the ratio $\frac{P^{(k)}(a)}{P'(a)}$ can be expressed in terms of the values at point a of polynomials u and v and their derivatives up to order $k-2$.

So, if one chooses f in CP_m or CP_m^+, one can apply Proposition 2.2 to get an element g in CP_n or CP_n^+ and then use Proposition 3.2 with g to get a polynomial which assumes a nonnegative value at the largest root of P.

Here is, as an example, the case dealt with in Laguerre's note. Consider the n-th Hermite polynomial H_n. It satisfies the differential equation $H_n'' - xH_n' + nH_n = 0$. If a is its largest root, one has

$$\frac{H_n''(a)}{H_n'(a)} = a \quad \text{and} \quad \frac{H_n^{(3)}(a)}{H_n'(a)} = a^2 + 1 - n.$$

Take $m = 2$ and $f = \sigma_1^2 - 4\sigma_2$. As already seen, the polynomial $(n-2)\sigma_1^2 - 2(n-1)\sigma_2$ is in $\mathrm{CP}_{(n-1)}$, so

$$(n-2)\left(\frac{a}{2}\right)^2 - 2(n-1)\frac{a^2+1-n}{6} = \frac{-(n+2)a^2 + 4(n-1)^2}{12} \geqslant 0.$$

Therefore, the polynomial $-(n+2)x^2 + 4(n-1)^2$ is nonnegative at the largest root of H_n. This gives the upper bound $\frac{2(n-1)}{\sqrt{n+2}}$ for the zeroes of H_n, already given in [1].

If we take $m = 3$ and $f = \sigma_1\sigma_2 - 9\sigma_3$, we get that the polynomial $x((4n-3)(n-1) - (n+3)x^2)$ is nonnegative at the largest root of H_n. This means that this largest root is less than $\sqrt{\frac{(4n-3)(n-1)}{n+3}}$, which improves on Laguerre's bound.

Here are a few computations done by using computer algebra. The following polynomials

$$x\bigl(-\bigl(n^2+n-4\bigr)x^2 + 2(2n-3)(n-1)^2\bigr),$$
$$-\bigl(11n^3 - 3n^2 - 82n + 104\bigr)x^4 + 48(n-2)(n-1)^3x^2 - 16(n-3)(n-1)^4,$$

and

$$-\bigl(13n^4 - 22n^3 - 117n^2 + 398n - 344\bigr)x^5 + 32(2n-5)(n-1)^4x^3 - 48\bigl(n^2 - 5n + 5\bigr)(n-1)^4x$$

are obtained by starting with $\sigma_1^m - m^m\sigma_m$, for $m = 3, 4, 5$. They give successive improvements on the bound for the zeroes of H_n.

As a curio, the last polynomial gives

$$2(n-1)\sqrt{\frac{4\bigl(2n^3 - 9n^2 + 12n - 5\bigr) + \sqrt{25n^6 - 315n^5 + 1890n^4 - 6395n^3 + 12501n^2 - 13050n + 5560}}{13n^4 - 22n^3 - 117n^2 + 398n - 344}}$$

as a bound for the largest root of H_n.

The polynomial

$$-\bigl(n^3 + 2n^2 + 15n + 18\bigr)x^6 + 6\bigl(2n^4 - 4n^3 + 5n^2 - 3n + 18\bigr)x^4$$
$$- 3\bigl(16n^3 - 64n^2 + 33n + 126\bigr)(n-1)^2x^2 + 64(n-3)^2(n-1)^4$$

is obtained by taking for f the discriminant of $x^3 - \sigma_1x^2 + \sigma_2x - \sigma_3$, but this does not improve on previous bounds.

References

[1] E. Laguerre, Sur les équations algébriques dont le premier membre satisfait à une équation linéaire du second ordre, C. R. Acad. Sci. Paris XC (1880) 809–812.