Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 347 (2009) 395–400

COMPTES RENDUS MATHEMATIQUE

Dynamical Systems

Dimension and measure for semi-hyperbolic rational maps of degree 2

Magnus Aspenberg [1], Jacek Graczyk

Laboratoire de mathématique, Université de Paris-Sud, 91405 Orsay cedex, France

Received 27 May 2008; accepted 11 February 2009

Available online 17 March 2009

Presented by Étienne Ghys

Abstract

We prove that almost every non-hyperbolic rational map of degree 2 has at least one recurrent critical point. This estimate is optimal because the set of rational maps with all critical points non-recurrent is of full Hausdorff dimension. ***To cite this article: M. Aspenberg, J. Graczyk, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Dimension et mesure pour les applications rationnelles semi-hyperboliques de degré 2. Nous démontrons que presque toute application rationnelle semi-hyperbolique de degré 2 a au moins un point critique récurrent. Cette estimation est optimale parce que l'ensemble des applications rationnelles avec tous les points critiques non-récurrents est de pleine dimension de Hausdorff. ***Pour citer cet article : M. Aspenberg, J. Graczyk, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soit $J(f)$ l'ensemble de Julia de l'application rationnelle f et $\mathrm{Crit}(f)$ l'ensemble des points critiques de f. L'ensemble oméga limite de x est défini par $\omega(x) = \{z \in \hat{\mathbb{C}}\colon\ \exists n_j\ f^{n_j}(x) \to z\}$. Une application rationelle est dite *hyperbolique* si $\omega(c) \cap J(f) = \emptyset$ pour tout $c \in \mathrm{Crit}(f)$.

Définition 0.1. Soit f une application rationnelle non-hyperbolique sans points périodiques paraboliques. On dit que f est *semi-hyperbolique* si $c \notin \omega(c)$ pour tout $c \in \mathrm{Crit}(f) \cap J(f)$. L'application f est de type *Misiurewicz* si $\omega(c) \cap \mathrm{Crit}(f) = \emptyset$ pour tout $c \in \mathrm{Crit}(f) \cap J(f)$.

L'espace $\mathcal{R}_d$ des applications rationnelles de degré $d \geqslant 2$ muni de la topologie de la convergence uniforme est une variété complexe de dimension $2d + 1$. Le plongement naturel de $\mathcal{R}_d$ dans $\mathbb{P}(\mathbb{C}^{2d+1})$ est un ouvert dense.

E-mail address: jacek.graczyk@math.u-psud.fr (J. Graczyk).

[1] The first author was funded by the Research Training Network CODY and SPADE2 of the European Commission.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.02.016

Nous démontrons deux théorèmes :

Théorème 0.2. *L'ensemble des applications semi-hyperboliques dans $\mathcal{R}_2$ est de mesure de Lebesgue nulle.*

Théorème 0.3. *L'ensemble des applications semi-hyperboliques dans $\mathcal{R}_d$, $d \geqslant 2$, est de dimension de Hausdorff $4d+2$.*

Dans la théorie des systèmes dynamiques, les applications rationnelles conjuguées par une application de Möbius sont souvent identifiées. On considère donc plutôt l'espace $[\mathcal{R}_d]$ des classes de conjugaison de Möbius des applications rationnelles de degré d. L'espace $[\mathcal{R}_d]$ est une variété complexe de dimension $2d-2$. Observons que la propriété de Misiurewicz ou de semi-hyperbolicité est un invariant topologique. Comme corollaire immédiat du Théorème 0.2, nous obtenons que la mesure de Lebesgue des classes de conjugaison semi-hyperboliques dans $\mathcal{R}_2$ est nulle.

On commence avec l'observation simple qu'il suffit de montrer que l'ensemble des applications *δ-semi-hyperboliques* n'a pas de points de densité de Lebesgue (on dit que f est δ-semi-hyperbolique si $\omega(c) \notin B(c,\delta)$ pour tout $c \in \mathrm{Crit}(f)$). Ensuite, le Théorème 0.2 est une conséquence immédiate du théorème suivant :

Théorème 0.4. *Soit $\delta > 0$ et f_a, $a \in B(0,r)$ une famille analytique d'applications rationnelles dans $[\mathcal{R}_2]$ avec f_0 une application δ-semi-hyperbolique qui n'est pas de type Misiurewicz. S'il existe f_a qui n'est pas Möbius conjuguée à f_0, alors la densité de Lebesgue des applications δ-semi-hyperboliques en $a=0$ est strictement inférieure à* 1.

Dans [1], il est demontré que l'ensemble des applications de Misiurewicz est de mesure de Lebesgue zéro dans $[\mathcal{R}_d]$ pour tout $d \geqslant 2$. Dans le cas de Misiurewicz l'ensemble $\omega(c)$ est disjoint de $\mathrm{Crit}(f)$ pour tout $c \in \mathrm{Crit}(f) \cap J(f)$. Soit $P^k(f,c) = \bigcup_{n>k} f^n(c)$. Par le théorème de Mané [6], l'ensemble $P^k(f) = \bigcup_{c \in \mathrm{Crit}(f)\cap J(f)} P^k(f,c)$ est hyperbolique pour tout k assez grand si et seulement si f est de type Misiurewicz. Donc, dans le cas semi-hyperbolique, $P^k(f)$ n'est pas nécessairement hyperbolique. Néanmoins, par un simple argument combinatoire, il y a au moins un point critique c tel que $f(c)$ appartienne à un ensemble hyperbolique Λ.

Par la théorie générale des ensembles hyperboliques, il existe un mouvement holomorphe de Λ : $h : \mathbb{B}(0,r) \times \Lambda \to \hat{\mathbb{C}}$, où $\mathbb{B}(0,r)$ est une boule paramétrique de dimension complexe $2d+1$ et $f = f_0$ est de type Misiurewicz ou semi-hyperbolique. Alors, $h_a(z)$ est injectif sur Λ et $h_z(a)$ est holomorphe dans $\mathbb{B}(0,r)$. Si f est de type Misiurewicz on peut alors définir les fonctions $x_j(a) = \theta_j(a) - h_a(\theta_j(0))$, où $\theta_j(0) = f^{k_j}(c_j(0)) \in \Lambda$, $j = 1,2$ et $c_j = c_j(0)$ sont les points critiques dans $J(f)$. Par [1] on peut supposer que f est semi-hyperbolique mais pas de type Misiurewicz. Par conséquent, on a seulement une fonction $x_j(a)$ bien définie, disons $x_1(a)$. Si x_j n'est pas identiquement nulle, on dit que x_j est *transverse*.

La fonction x_1 est en effet transverse dans presque tout disque $B_v(0,r) \subset \mathbb{B}(0,r)$ de dimension complexe 1 paramétrisé par directions $v \in \mathbb{P}(\mathbb{C}^4)$, voir le Lemme 2.9. On montre le lemme par l'absurde. Supposons que x_1 soit égal à zéro pour tout $a \in B_v(0,r)$ et que la famille f_a, $a \in B_v(0,r)$, ne soit pas une famille de conjugaisons de Möbius de f_0. Selon de Théorème 4.2 de [5], l'une des deux fonctions $\xi_{n,j}(a) = f_a^n(\theta_j(a))$ n'est pas normale dans $B_v(0,r)$. Par notre hypothèse que x_1 n'est pas transverse, $\xi_{n,1}(a)$ appartient à $h_a(\Lambda) = \Lambda_a$ pour tout $a \in B_v(0,r)$ et donc $\xi_{n,1}$ est normale sur $B_v(0,r)$. Par conséquent, il faut que $\xi_{n,2}$ ne soit pas normale dans $B_v(0,r)$ et donc on trouve un paramètre $a_0 \in B_v(0,r)$ tel que $\xi_{n,2}(a_0) = c_2(a_0)$. L'application f_{a_0} est de type Misiurewicz et donc il y a une composante structurellement stable d'applications de Misiurewicz dans $\mathbb{B}(0,r)$, une contradiction. Donc, x_1 est transverse.

Pour démontrer le Théoreme 2.3, on utilise la Proposition 4.3 de [1] qui est formulée comme le Théorème 2.10. Le Théorème 0.4 découle du Théorème 2.3.

Pour démontrer le Théorème 0.3, on va s'appuyer sur [9]. On prend une famille $f_{\varepsilon,\lambda}(z) = \varepsilon z^d + z^2 + \lambda$ avec $\varepsilon > 0$ assez petit pour que tous les points critiques sauf $z=0$ appartiennent au bassin attractif de ∞. On note $f_\lambda := f_{\varepsilon,\lambda}$. Il existe un $b > 0$ tel que si $\lambda \in B(0,b)$ on a $B(0,10) \Subset f_\lambda(B(0,10))$ et f est propre de degré 2 dans $B(0,10)$. La famille f_λ, $\lambda \in B(0,b)$, est donc d'allure quadratique. Soit $\mathcal{M}$ l'ensemble de Mandelbrot. Par la théorie des fonctions d'allure quadratique, [3], pour tout $c \in \partial\mathcal{M}$ il existe un paramètre $\lambda = \lambda(c)$ tel que f_λ et $p_c(z) = z^2 + c$ sont conjugués par un homéomorphisme quasiconforme.

Rappelons que la dimension hyperbolique $\mathrm{Dim}_{\mathrm{hyp}}(J)$ d'un ensemble de Julia J, voir (1), est le supremum de la dimension de Hausdorff des sous-ensembles hyperboliques de J. Selon [9], il existe $c_0 \in \partial\mathcal{M}$ tel que

$\mathrm{Dim}_{\mathrm{hyp}}(J(p_{c_0})) = 2$. Lorsque tous les points critiques de f_λ sauf $z = 0$ appartiennent au bassin de ∞, on peut utiliser le Théorème 3.2. L'application $z^2 + c_0$ n'est pas structurellement stable et donc on obtient (3), $\mathrm{Dim}_H(\{\lambda \in \Lambda \cap U\colon\ f_\lambda \text{ est Misiurewicz}\}) \geqslant \mathrm{Dim}_{\mathrm{hyp}}(f_{\lambda_0})$. On conclut que dans toute famille analytique contenant f_λ, l'ensemble des applications semi-hyperboliques est de dimension de Hausdorff 2. Le Théorème 0.3 en découle par la formule du produit.

1. Introduction

The space of rational maps $\mathcal{R}_d$ of the degree d with the topology of uniform convergence on the Riemann sphere $\hat{\mathbb{C}}$ is a complex manifold of dimension $2d + 1$. A natural embedding of $\mathcal{R}_d$ in the projective space $\mathbb{P}(\mathbb{C}^{2d+1})$ is open and dense. Indeed, every rational map $f \in \mathcal{R}_d$ is determined up to a constant multiple by their d poles and d zeros. In the theory of dynamical systems rational functions conjugate by a Möbius transformation are often identified. This allows one to reduce 3 parameters and the space $[\mathcal{R}_d]$ of Möbius conjugacy classes of rational maps of degree d is an open manifold of complex dimension $2d - 2$.

Let $\mathrm{Crit}(f)$ be the set of critical points for f, $J(f)$ the Julia set of f and $F(f)$ the Fatou set of f. Recall that the omega-limit set $\omega(c)$ of some point c is the set of limit points in the set of forward orbits of c. A rational map f is named *hyperbolic* if $\omega(c) \cap J(f) = \emptyset$ for every critical point $c \in \mathrm{Crit}(f)$.

Our Theorem that non-hyperbolic rational maps with at least one critical point recurrent are of full Lebesgue measure should be true for any degree $d > 2$ but we have no proof. Another interesting question is whether the set of non-hyperbolic rational maps of degree 2 with one critical point of the "Misiurewicz type" is of Lebesgue measure zero or not. We say that a critical point c of a rational map f is of *the Misiurewicz type* if $\omega(c)$ stays at the positive distance to $\mathrm{Crit}(f)$. To approach this problem using a general strategy presented in this paper one will have to first prove that every non-hyperbolic rational of degree 2 with at least one critical point of the Misiurewicz type is not structurally stable.

2. Semi-hyperbolic rational maps of degree 2 are rare

Definition 2.1. A non-hyperbolic rational map is *semi-hyperbolic* is it has no parabolic periodic points and such that for every $c \in \mathrm{Crit}(f) \cap J(f)$, we have $c \notin \omega(c)$. A rational map is called a Misiurewicz map if it is semi-hyperbolic and every critical point $c \in J(f)$ is of the Misiurewicz type.

Theorem 2.2. *The set of semi-hyperbolic maps in $\mathcal{R}_2$ is of 10-dimensional Lebesgue measure zero.*

Observe that semi-hyperbolic and Misiurewicz maps are topologically invariant. In particular, any semi-hyperbolic map (Misiurewicz) remains so after Möbius change of coordinates in the Riemann sphere. Therefore, one can study semi-hyperbolic (Misiurewicz) Möbius conjugacy classes in the space $[\mathcal{R}_2]$. As a corollary to Theorem 2.2, we have that the set of semi-hyperbolic conjugacy classes in $[\mathcal{R}_2]$ is of 4-dimensional Lebesgue measure zero.

Let $B(0,r)$ be a 1-dimensional parametric disk of radius $r > 0$ and let $\mathbb{B}(0,r)$ be a 5-dimensional disk. Moreover, each 1-dimensional disk $B_v(0,r) \subset \mathbb{B}(0,r)$ can be determined by a direction $v \in \mathbb{P}(\mathbb{C}^{2d})$ such that the plane in which $B_v(0,r)$ lies can be parameterized as $\{a \in \mathbb{C}\colon\ a = (t\alpha_1, t\alpha_2, \ldots, t\alpha_{2d+1}),\ t \in \mathbb{C}\}$ where $(\alpha_1, \ldots, \alpha_{2d+1})$ is a representative for $v \in \mathbb{P}(\mathbb{C}^{2d})$. We prove Theorem 2.2 by showing a one dimensional slice version for certain "good directions" as follows.

We say that f is *δ-semi-hyperbolic* iff f is semi-hyperbolic and $\omega(c) \cap B(c,\delta) = \emptyset$ for every $c \in J(f)$.

Theorem 2.3. *For almost all directions $v \in \mathbb{P}(\mathbb{C}^4)$ the following is true. Let f_a, $a \in B_v(0,r)$ be an analytic family of rational maps, where f_0 is semi-hyperbolic. For every $\delta > 0$, the Lebesgue density of the set of δ-semi-hyperbolic maps at $a = 0$ is strictly less than 1.*

Theorem 2.2 then easily follows from Theorem 2.3 and Fubini's Theorem.

Any rational map of degree 2 has 2 critical points including multiplicity. If only one critical point is in the Julia set then the map satisfies the Misiurewicz condition and Theorem 2.2 follows from an earlier work [1]. Without loss of generality, we may assume that f is semi-hyperbolic but not of the Misiurewicz type and its both critical points c_1, c_2 are in the Julia set.

The first observation we want to point out is that if $c_1 \in \omega(c_2)$ then we cannot have $c_2 \in \omega(c_1)$, because then c_2 would be recurrent. Hence we may assume that $c_2 \notin \omega(c_1)$. This means that the forward orbit of c_1 does not contain any critical point. Let us write $\Lambda = \overline{\bigcup_{n>0} f^n(c_1)}$.

Definition 2.4. A set X is a hyperbolic subset for f if $f(X) \subset X$ and there exists constants $N > 0$ and $\lambda > 1$ such that for each $x \in X$ we have

$$\left|(f^N)'(x)\right| \geqslant \lambda.$$

According to a Theorem of Mañé [6], the set Λ is hyperbolic. Hence there is a holomorphic motion on Λ;

$$h : \Lambda \times B(0, r) \to \hat{\mathbb{C}},$$

where $B(0, r)$ is a small parametric disk around the starting function $f = f_0$. Put $\xi_{n,j}(a) = f_a^n(\theta_j(a))$, where $\theta_j(a) = f_a(c_j(a))$, $j = 1, 2$.

Let us state a special case of Theorem 4.2 from [5], see also [7].

Theorem 2.5. *Let X be a connected complex manifold. Let f_λ be a holomorphic family of rational maps, parameterized by X. Suppose that the critical points $c_i(\lambda)$ are holomorphic functions of λ. Let x be a point in X. Then the following conditions are equivalent*:

- *The Julia set moves holomorphically at x.*
- *For each i, the functions $f_\lambda^n(c_i(\lambda))$ form a normal family at x.*

Theorem 2.6. *The Julia set of a semi-hyperbolic map has Lebesgue measure zero unless it is the whole Riemann sphere.*

Proof. For polynomials, the theorem is proved in [2]. The general case follows from the recent work [8]. □

Lemma 2.7. *If f is semi-hyperbolic, $J(f) = \hat{\mathbb{C}}$ and $J(f)$ carries an invariant line field, then f has to be a Lattés map.*

Proof. The main line of the argument follows [5]. Assume that E is the support of the invariant line field, and that E has positive measure. Take a density point z_0 of E such that the line field Θ is almost continuous at z_0.

Now take a limit point x of $\{f^n(z_0)\colon n \geqslant 0\}$. Hence there is a subsequence n_k such that $f^{n_k}(z_0) \to x$. Choose the preimages $W_k = f^{-n_k}(B(x, \eta))$ such that $z_0 \in W_k$. By Mañé's Theorem there is a ball $B(x, \eta)$, where $\eta > 0$, such that $\deg(f^{n_k}|_{W_k}) \leqslant N$ for all $k \geqslant 0$, where $N < \infty$ is a constant and such that $\operatorname{diam}(W_k) \to 0$. Since z_0 is a density point,

$$\lim_{r\to 0} \frac{m(E \cap B(z_0, r))}{m(B(z_0, r))} \to 1.$$

Now we use Lemma 2.2 in [2], to get that for every $C' > 0$ there is some constant q only depending on N and C' such that

$$f^{n_k}\left(\left\{w \in W_k\colon \rho_{W_k}(z, z_0) \leqslant C'\right\}\right) \supset \left\{z \in B(x, \eta)\colon \rho_{B(x,\eta)}\left(z, f^{n_k}(z_0)\right) \leqslant q\right\}.$$

Put $W_k' = \{w \in W_k\colon \rho_{W_k}(z, z_0) \leqslant C'\}$. Choosing C' sufficiently small and using Teichmuller's modulus theorem (see for instance [4]), we see that there is a ball B_k with the boundary contained in $W_k \setminus W_k'$ and $\operatorname{mod}(W_k \setminus B_k) \geqslant 1$.

Let $A_k : B_k \to D$ be a linear normalization of B_k, where D is the unit disk. Then $f_k = f^{n_k} \circ A_k^{-1}$ is a normal family on an open set $D' \Supset D$ and $f_k(D) \supset \{z \in B(x, \eta)\colon \rho_{B(x,\eta)}(z, f^{n_k}(z_0)) \leqslant q\}$. Hence there is a subsequence k' such that $f_{k'}$ converges uniformly to a non-constant limit function f. Since z_0 is a Lebesgue point for Θ, the family of line fields $(A_k)_*(\Theta)$ tends to a constant line field Θ' in D. Since f is of bounded degree, it is obvious that $f_*(\Theta')$ is a holomorphic line field on an open subset of $f(D)$. By invariance, $f_*(\Theta')$ coincides with Θ on $f(D)$. It now follows from Lemma 3.16 in [5] that f is a Lattés map. □

Definition 2.8. Given a critical value $\theta_j(a) = f_a(c_j(a))$, we introduce the parameter function

$$x_j(a) = \theta_j(a) - h_a\big(\theta_j(0)\big).$$

We say that $x_j(a)$ is *transversal* in some set B if it is not identically equal to zero in B. If it is clear in which set $x_j(a)$ is transversal in, we just say x_j is transversal.

Recall that the orbit of $c_1(a)$ lands at the holomorphically moving hyperbolic set Λ_a.

Lemma 2.9. *Let $B_v(0,r)$ be a 1-dimensional ball in $\mathbb{B}(0,r)$ with direction $v \in \mathbb{P}(\mathbb{C}^4)$. Then the set of directions v where the function $x_1(a) = \theta_1(a) - h_a(\theta_1(0))$ is transversal in $B_v(0,r)$ has full Lebesgue measure.*

Proof. We will show that the set $\{a \in \mathbb{B}(0,r)\colon\ x_1(a) = 0\}$ is an analytic set of codimension 1.

In order to reach a contradiction, suppose that $x_1(a) = 0$ for every a in $\mathbb{B}(0,r)$. Lemma 2.7 and Theorem 2.6 imply that the Julia set of f_0 carries no invariant line field. By Theorem 2.5, the functions $\xi_{n,j}(a)$ cannot be both normal in any $B(0,r) = B_v(0,r) \subset \mathbb{B}(0,r)$ unless f_a, $a \in B(0,r)$, is a family of Möbious conjugations of f_0. Without loss of generality, we may assume that the latter is not the case because the set of all Möbius conjugations of f_0 is of complex dimension 3. From our hypothesis, $\xi_{n,1}(a)$ is normal. Thus $\xi_{n,2}(a)$ is not normal. Then for some n the set $\xi_{n,2}(B(0,r))$ will cover the whole Riemann sphere minus at most 2 points. In particular, three preimages of the critical point $c_2(a)$ under f_a cannot be avoided by $\xi_n(B(0,r))$. This means that there is a solution to $\xi_{n,2}(a_0) = c_2(a_0)$ for some $a_0 \in B(0,r)$. This implies that there is a structurally stable component in $\mathbb{B}(0,r)$ of Misiurewicz maps, a contradiction. Now, the claim about the codimension 1 follows from the well-known theory of [7]. □

Now we are ready to complete the proof of Theorem 2.2.

Theorem 2.10. *Let f_a, $a \in B = B(0,\varepsilon)$, $\varepsilon > 0$, be an analytic family of rational maps. Assume that some critical value $\theta(0)$ of f_0 belongs to a hyperbolic set Λ and that the transversality criterium is satisfied, i.e. the function $x(a) = \theta(a) - h_a(\theta(0))$ is not a constant function equal to 0.*

Let $\xi_n(a) := f_a^n(\theta(a))$. Then for every $0 < r < \varepsilon$ small enough there is $n(r) > 0$ such that the family defined in the unit disk D,

$$\xi_{n(r)}(rz) : D \to \mathbb{C}$$

is normal. Moreover, every limit function is non-constant.

Proof. Theorem 2.10 is a direct consequence of Proposition 4.3 in [1]. Namely, it is proved that for every r small enough, there is $n(r)$ such that the function $F_r(z) = \xi_{n(r)}(rz)$ maps the unit disk onto a set contained in a disk of diameter $\delta' > 0$. This implies that the family $F_r(z) = \xi_{n(r)}(rz) : D \to \mathbb{C}$ is normal. By the same Proposition 4.3, F_r has bounded distortion on some annulus $A = \{1/10 < |z| < 1\}$. This implies that the image $F_r(D')$ of $D' = \{|z| \leqslant 1/2\}$ also contains a disk of some fixed diameter. Hence every limit function is not a constant. □

Suppose that f_0 is the Lebesgue density point of δ- semi-hyperbolic maps. Without loss of generality, f_0 is also a δ-semi-hyperbolic map. Let f_a, $a \in B_v(0,\varepsilon)$, $v \in \mathbb{P}(\mathbb{C}^4)$ be an analytic family as in Theorem 2.3. Then for almost all $v \in \mathbb{P}(\mathbb{C}^4)$, the function $x_1(a) = \theta_1(a) - h_a(\theta_1(a))$ is transversal, see Lemma 2.9. By Theorem 2.10, choose a converging sequence $\varphi_i(z) := \xi_{n(r_i)}(r_i z) : D \to \mathbb{C}$ and suppose that $\varphi : D \to \mathbb{C}$ is a limit function. Let A_i be a set of semi-hyperbolic parameters inside D. Then the area of $\xi_{n(r_i)}(A_i)$ tends to the area of $\varphi(D)$. By the eventually onto property, there exists N so that $f_0^N(\varphi(D))$ contains the Julia set $J(f_0)$. Consequently, $f_a^N \circ \xi_{n(r_i)}(A_i)$ becomes dense in $J(f_0)$ when i tends to ∞. On the other hand, for every $a \in A_i$, $f_a^N \circ \xi_{n(r_i)}(a)$ stays at the distance at least δ from c_1, a contradiction.

3. Full Hausdorff dimension for semi-hyperbolic maps

Theorem 3.1. *The Hausdorff dimension of Misiurewicz rational maps in $\mathcal{R}_d$, $d \geqslant 2$, is equal to $4d + 2$.*

Let Mis_d be the set of all Misiurewicz maps of degree d. Fix a degree $d \geqslant 2$. The idea is to find a Misiurewicz map of degree d with the Hausdorff dimension arbitrarily close to 2 and only one critical point in its Julia set. For this purpose, for $d \geqslant 3$ we consider the family of polynomials

$$f_{\varepsilon,\lambda}(z) = \varepsilon z^d + z^2 + \lambda,$$

for some small $\varepsilon \in \mathbb{C}$, $|\varepsilon| > 0$. If $d = 2$ then put $f_{\varepsilon,\lambda}(z) = z^2 + \lambda$. The critical points of $f_{\varepsilon,\lambda}$ are $0, \infty$ and, if $d \geqslant 3$, also the $(d-2)$th roots of $-2/(\varepsilon d)$.

It is clear that choosing $|\varepsilon|$ sufficiently small, there is some $b > 0$ such that the map $f_\lambda = f_{\varepsilon,\lambda}$ for $\lambda \in B(0, b)$, has the property that every critical point but possibly $z = 0$ lie in the immediate basin of attraction of ∞. In fact, we can ensure that $f_\lambda(B(0, 10)) \Supset B(0, 10)$ and that f_λ is a $2-1$ covering on $B(0, 10)$. Hence f_λ restricted to $B(0, 10)$ is a quadratic like map, for $\lambda \in B(0, b)$. The family f_λ is a Mandelbrot like family in the sense of [3]. Let $\mathcal{M}$ denote the usual Mandelbrot set, being the connectedness locus of the family $p_c(z) = z^2 + c$. By the theory of quadratic-like maps in [3] (see Example 4.1 in [3]), for every $c \in \partial\mathcal{M}$ there exists $\lambda(c)$ such that $f_{\varepsilon,\lambda}$ is quasiconjugate to $z^2 + c$.

Let us denote by $\mathrm{Dim}_H(E)$ the Hausdorff dimension of a set E and $\mathrm{Dim}_{\mathrm{hyp}}(E)$ the hyperbolic dimension for a set E. The *hyperbolic dimension* is defined as

$$\mathrm{Dim}_{\mathrm{hyp}}(f) = \sup\bigl\{\mathrm{Dim}_H(X)\colon\ X \text{ is a hyperbolic subset for } f\bigr\}. \tag{1}$$

By [9], there exists $c_0 \in \partial\mathcal{M}$ such that the $\mathrm{Dim}_{\mathrm{hyp}}(J(p_{c_0})) = 2$. We formulate verbatim Theorem 1 from [9].

Theorem 3.2. *Let f_λ be a complex 1-dimensional family of rational maps, such that f_{λ_0} is not structurally stable. Let Λ be the set of structurally unstable parameters in this family. Then if U is an open subset containing λ_0, we have*

$$\mathrm{Dim}_H\bigl(\{\lambda \in \Lambda \cap U\colon\ \textit{There is a hyperbolic subset for } f_\lambda \textit{ containing the forward orbit of a critical point}\}\bigr) \geqslant \mathrm{Dim}_{\mathrm{hyp}}(f_{\lambda_0}). \tag{2}$$

Note that the set on the left-hand side above is in general larger than the set of Misiurewicz maps in U. Since at most one critical point of the maps the family f_λ is in the Julia set, the condition in Theorem 3.2 becomes in fact equivalent to the Misiurewicz condition. Hence,

$$\mathrm{Dim}_H\bigl(\{\lambda \in \Lambda \cap U\colon\ f_\lambda \text{ is of the Misiurewicz type}\}\bigr) \geqslant \mathrm{Dim}_{\mathrm{hyp}}(f_{\lambda_0}). \tag{3}$$

By the choice of $c_0 \in \partial\mathcal{M}$, $f_{\varepsilon,\lambda_0}$ does not have an invariant line field and thus every analytic family of rational maps passing though f_{λ_0} is structurally unstable unless it is a family of Möbius reparameterizations ($M_c^{-1} \circ g \circ M_c = f$). Thus in any analytic (1-dimensional) family of rational maps passing through f_{λ_0} in $\mathcal{R}_d$, the set of Misiurewicz maps has Hausdorff dimension 2. Now by the product formula,

$$\mathrm{Dim}_H(\mathcal{M}_d) \geqslant \mathrm{Dim}_H\bigl(\mathbb{P}\bigl(\mathbb{C}^{2d}\bigr)\bigr) + \inf_{v \in \mathbb{P}(\mathbb{C}^{2d})} \mathrm{Dim}_H\bigl(\mathcal{M}_d \cap B_v(c_0, r)\bigr) = 4d + 2.$$

Acknowledgements

We would like to thank Duncan Sands for many helpful discussions about semi-hyperbolic maps.

References

[1] M. Aspenberg, Rational Misiurewicz maps are rare II, Preprint, math.DS/0703306, Comm. Math. Phys., in press.
[2] L. Carleson, P.W. Jones, J.-C. Yoccoz, Julia and John, Bol. Soc. Brasil. Mat. (N.S.) 25 (1994) 1–30.
[3] A. Douady, J. Hubbard, On the dynamics of polynomial-like mappings, Ann. Sci. École Norm. Sup. 18 (1985) 287–343.
[4] O. Lehto, K.I. Virtanen, Quasiconformal Mappings in the Plane, Springer-Verlag, New York, 1973, viii+258.
[5] C.T. McMullen, Complex Dynamics and Renormalization, Annals of Mathematics Studies, vol. 135, Princeton Univ. Press, Princeton, NJ, 1994, x+214.
[6] R. Mañé, On a theorem of Fatou, Bol. Soc. Brasil. Mat. (N.S.) 24 (1993) 1–11.
[7] R. Mañé, P. Sad, D. Sullivan, On the dynamics of rational maps, Ann. Sci. École Norm. Sup. 16 (1983) 193–217.
[8] N. Mihalache, La condition de Collet–Eckmann pour les orbites critiques recurrents, Ergod. Th. Dynam. Sys. 27 (4) (2007) 1267–1286.
[9] M. Shishikura, The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets, Ann. of Math. (2) 147 (1998) 225–267.