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Ordinary Differential Equations

Existence of periodic solutions of a ordinary differential equation perturbed by a small parameter: An averaging approach

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Abstract

In this Note, we obtain two new results for existence of periodic solutions for differential equations perturbed by a small parameter. The first one is based on a new fixed point theorem previously obtained by the authors. The second one is based on study of suitable linearized equations. Our approach deals with degree theory and nonsmooth analysis. ***To cite this article: A. Gudovich et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sur l'existence de solutions périodiques pour une équation différentielle perturbée par un petit paramètre : une approche par moyennisation. Dans cette Note, nous donnons deux résultats nouveaux sur l'existence de solutions périodiques pour des équations différentielles perturbées par un petit paramètre. Le premier résultat est lié à un nouveau théorème de point fixe précédemment obtenu par les auteurs ; le second est déduit de l'étude d'une équation différentielle linéarisée. Cette approche relève de la théorie du degré topologique et de l'analyse non régulière. ***Pour citer cet article : A. Gudovich et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Il s'agit d'étudier l'existence de solutions T-périodiques de l'équation différentielle

$$x'(t) = \varepsilon f\big(t, x(t)\big), \quad t \geqslant 0 \tag{1}$$

quand le paramètre $\varepsilon > 0$ est suffisamment petit. Ici $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ est continue et T-périodique par rapport à la première variable. A l'équation précédente on associe le problème auxilliaire suivant :

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$$x'(t) = \varepsilon f_0\big(x(t)\big), \quad \text{avec } f_0(\xi) = \frac{1}{T}\int_0^T f(s,\xi)\,\mathrm{d}s. \tag{2}$$

Nous déduirons l'existence de solutions périodiques, de l'existence d'un ensemble K pour lequel le degré topologique $\deg(-f_0, K)$ de f_0 sur K sera non nul. Pour estimer $\deg(-f_0, K)$, nous allons à la fois appliquer des méthodes topologiques décrites dans [1,10,11,17] et utiliser un résultat récent [9] d'existence de point fixe dans un ensemble épilipschitzien borné. Dans toute la suite l'ensemble K sera supposé épilischitzien [14] c'est à dire que son épigraphe est localement l'épigraphe d'une fonction lipschitzienne (voir dans [3,14,4] des caractérisations équivalentes). Rappelons qu'un ensemble fermé épilipchitzien est nécessairement l'adhérence d'un ensemble ouvert et que donc le degré topologique est bien défini. La classe des ensembles épilipchitziens est assez vaste car elle contient tous les convexes et les ensembles à bord C^1.

Notre approche se fonde sur une relation entre le degré topologique et les caractéristiques d'Euler d'ensembles définis par des propriétés des trajectoires d'équations différentielles. Considérons l'équation différentielle suivante :

$$x'(t) = g\big(x(t)\big), \quad x(0) = x_0 \tag{3}$$

où $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ est continue. Appelons $S_g(x_0)$ l'ensemble des solutions absoluement continues de (3). Nous définissons maintenant un sous-ensemble du bord de K (qui apparait naturellement dans la formulation du principe topologique de Wazewski [5]) :

$$K^s(g) = \big\{x_0 \in \partial K,\ \forall x(\cdot) \in S_g(x_0),\ \exists \sigma > 0\colon\ x\big((0,\delta]\big) \cap K = \emptyset\big\}.$$

L'ensemble $K^s(g)$ peut être vu comme l'ensemble des points $x_0 \in \partial K$ du bord de K à partir desquels toutes les trajectoires sortent immédiatement de K. On note par $\chi(K)$ la caractéristique d'Euler de K (voir par exemple [16]).

Rappelons maintenant le résultat suivant qui est une légère extension de la Proposition 4.1 de [9] :

Lemme 0.1. *Supposons que $g(x) \neq 0$ pour tout $x \in \partial K$, que $\chi(K^s(f_0))$ soit bien défini et que l'ensemble $K^s(g)$ soit fermé. Alors*

$$\deg(-g, K) = \chi(K) - \chi\big(K^s(g)\big). \tag{4}$$

Notre premier résultat principal est alors le suivant :

Théorème 0.2. *Supposons que $K^s(f_0)$ soit fermé et que la caractéristique d'Euler $\chi(K^s(f_0))$ soit bien définie. Si f_0 ne s'annule pas sur le bord de K et que $\chi(K) \neq \chi(K^s(f_0))$, alors pour tout $\varepsilon > 0$, assez petit, le système (1) a au moins une solution T-périodique $x_\varepsilon \in C_T(K)$. De plus pour toute suite $\varepsilon_n \to 0$, la suite des solutions x_{ε_n} est relativement compacte et chacune de ses valeurs d'adhérence est une constante x_* vérifiant $f_0(x_*) = 0$.*

Notre second résultat concerne l'existence de solution périodiques de

$$x'(t) = \varepsilon\phi\big(t, x(t)\big) + \psi\big(t, x(t)\big), \tag{5}$$

où $\phi, \psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ est continue et T-périodique. Suivant [15], après le changement de variable $z(t) = \Omega(0, t, x(t))$ (où $\Omega(\cdot, t_0, \xi)$ désigne la solution de (5) où $\varepsilon = 0$ et avec la condition initiale $x(t_0) = \xi$), le système (5) peut se mettre sous la forme de (1). Il est possible d'étudier l'existence de trajectoire périodique de (5) sans supposer que le changement de variable précédent soit T-périodique, pour cela on considère le système linéarisé suivant :

$$y' = \frac{\partial\psi}{\partial x}\big(t, \Omega(t,0,\xi)\big)y + \phi\big(t, \Omega(t,0,\xi)\big), \tag{6}$$

où $\xi \in \mathbb{R}^n$. On note par $\eta(\cdot, s, \xi)$ la solution de (12), vérifiant $y(s) = 0$ on définit $\bar{f}_0(\xi) = \eta(T, 0, \xi)$. D'après [7], s'il existe un ouvert borné $U \subset \mathbb{R}^n$ vérifiant $\deg(\bar{f}_0, U) \neq 0$ alors pour tout $\varepsilon > 0$ assez petit, il existe une trajectoire T-périodique de (5). Grâce à cela nous avons pu obtenir le résultat suivant :

Théorème 0.3. *Supposons que $\phi, \psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ soit de classe C^1, que K soit un ensemble épilipchitzien compact, que $K^s(\bar{f}_0)$ soit fermé et que sa caractéristique d'Euler soit bien définie. Si $\bar{f}_0$ ne s'annule pas sur le bord de K, si $\chi(K) \neq \chi(K^s(\bar{f}_0))$ et si les deux conditions suivantes sont satisfaites*

$$\text{(A0)} \quad \Omega(T, 0, \xi) = \xi, \quad \xi \in \partial K,$$

$$\text{(A1)} \quad \eta(T, s, \xi) - \eta(0, s, \xi) \neq 0, \quad s \in [0, T],\ \xi \in \partial K,$$

alors il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in [0, \varepsilon_0]$ le système (5) *a au moins une solution T-périodique vérifiant $\Omega(0, t, x_\varepsilon(t)) \in K$, $t \in [0, T]$.*

1. Introduction

We study the problem of the existence and dependence on a small parameter $\varepsilon > 0$ of T-periodic solutions of the following differential equation:

$$x'(t) = \varepsilon f\big(t, x(t)\big), \quad t \geqslant 0 \tag{7}$$

where the function $f : R \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and T-periodic with respect to the first variable. We will use topological methods and degree theory (see, for instance, [1,10,11,17]). For this aim we consider the following auxiliary system:

$$x'(t) = \varepsilon f_0\big(x(t)\big), \quad \text{with } f_0(\xi) = \frac{1}{T}\int_0^T f(s, \xi)\,\mathrm{d}s. \tag{8}$$

Our main aim is to obtain the existence of T-periodic solutions of the system (7). We will deduce this from the existence of a compact set $K \subset \mathbb{R}^n$ such that for every point x in the boundary ∂K of K we have $f_0(x) \neq 0$ and such that the topological degree of f on K $\deg(-f_0, K)$ is not equal to zero. To estimate $\deg(-f_0, K)$, we need a precise information on the behavior of the trajectories of stationary equation (8) at $\varepsilon = 1$. The starting point of this approach is the Note [9] devoted to existence of the fixed point of the continuous functions on compact epilipschitz subsets of $\mathbb{R}^n$. In [9] a relation was established between the existence of equilibria of function f and the properties of the flows generated by trajectories of corresponding stationary differential equation (cf. also [5,6,12,13] for related questions).

Our second main problem concerns the existence of periodic solutions to the following differential equation:

$$x'(t) = \varepsilon\phi\big(t, x(t)\big) + \psi\big(t, x(t)\big), \tag{9}$$

where the functions $\phi, \psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable and T-periodic with respect to t. Systems of the form (9) represent a classical topic of the theory of differential equations depending on a small parameter and such systems have been studied by several different methods. In [15], a transformation of the considered system (9) into the standard form allows to apply the classical averaging principle (see, for instance, [2]). Namely, after the change of variable

$$z(t) = \Omega\big(0, t, x(t)\big), \tag{10}$$

where $\Omega(\cdot, t_0, \xi)$ denotes the solution x of

$$x'(t) = \psi\big(t, x(t)\big), \tag{11}$$

with the initial condition $x(t_0) = \xi$. The system (9) can be reduced to the form (7). In this case it is necessary to assume that the change of variable (10) is T-periodic for every T-periodic function x.

An another approach to study the periodic problem for Eq. (9), proposed in [7,8] does not require T-periodic condition for (10). This approach is based on the linearized system

$$y' = \frac{\partial\psi}{\partial x}\big(t, \Omega(t, 0, \xi)\big)y + \phi\big(t, \Omega(t, 0, \xi)\big), \tag{12}$$

where $\xi \in \mathbb{R}^n$. Denote by $\eta(\cdot, s, \xi)$ be the solution y of system (12), satisfying $y(s) = 0$ and define $\bar{f_0}(\xi) = \eta(T, 0, \xi)$. It was shown in [7] that the existence of an bounded open set $U \subset \mathbb{R}^n$ satisfying $\deg(\bar{f_0}, U) \neq 0$ provides for ε small enough the existence of T-periodic solutions of the system (9).

In the present paper we study the properties of trajectories of Eq. (8) with $f = \bar{f_0}$, $\varepsilon = 1$ and initial condition $x \in \partial K$ providing existence of T-periodic solutions of the system (9).

2. Preliminaries

Let $K \subset \mathbb{R}^n$ is an compact set. We assume that the set K is epilipschitz i.e. its boundary is locally homeorphic to the epigraph of a Lipschitz functions. The reader can find in [3,14,4] other characterizations an properties of compact epilipschitz sets. In particular we know that K being epilipschitz, it is the closure of a open set U and so its topological degree is well-defined. Let $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. We denote by $S_g(x_0)$ the set of absolutely continuous solutions to the Cauchy problem

$$x'(t) = g\big(x(t)\big), \quad t \geqslant 0, \quad x(0) = x_0. \tag{13}$$

Let us define the following subset of K which appears in a natural way in the Wazewski topological principle [5]:

$$K^s(g) = \big\{x_0 \in \partial K,\ \forall x(\cdot) \in S_g(x_0),\ \exists \sigma > 0\colon\ x\big((0, \delta]\big) \cap K = \emptyset\big\}.$$

That is $K^s(g)$ is the set of elements $x_0 \in \partial K$ such that the all solutions to the Cauchy problem (13) leave K immediately. We denote the Euler characteristic of the set K by $\chi(K)$ (for definition of the Euler characteristic see for instance [16]).

We will need the lemma which is a reformulation of Proposition 4.1 in [9]:

Lemma 2.1. *Let $K \subset \mathbb{R}^n$ be epilipschitz compact set and let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that $g(x) \neq 0$ for $x \in \partial K$. Suppose that $\chi(K^s(g))$ is well defined and the set $K^s(g)$ is closed. Then*

$$\deg(-g, K) = \chi(K) - \chi\big(K^s(g)\big). \tag{14}$$

3. Main results

We denote by $C([0, T], K)$ (respectively by $C_T(K)$) the set of all continuous (respectively continuous T-periodic) functions with values in K.

Theorem 3.1. *Let $K \subset \mathbb{R}^n$ be epilipschitz compact set. Assume that the set $K^s(f_0)$ is closed and the Euler characteristic $\chi(K^s(f_0))$ is well defined. If $f_0(x) \neq 0$ for every $x \in \partial K$ and $\chi(K) \neq \chi(K^s(f_0))$ then, for sufficiently small $\varepsilon > 0$, system (7) has at least one T-periodic solution $x_\varepsilon \in C_T(K)$. Moreover, for every sequence $\varepsilon_n \to 0$, as $n \to \infty$, the sequence x_{ε_n} is relatively compact and every limit point x_* of this sequence is a constant function such that $f_0(x_*) = 0$.*

Proof of Theorem 3.1. We define, for $\varepsilon > 0$ given, an integral operator $P_\varepsilon : C([0, T], K) \to C([0, T], \mathbb{R}^n)$ as follows:

$$P_\varepsilon x(t) = x(T) + \varepsilon \int_0^t f\big(s, x(s)\big)\,\mathrm{d}s, \quad t \in [0, T].$$

Note that from the continuity of the function f and from the boundedness of the set K it follows that for $\varepsilon > 0$ the operator P_ε is continuous and compact. It is easily seen that x_ε is a fixed point of P_ε, if and only if the T-periodic extension $\tilde{x}_\varepsilon$ of the function x_ε is a T-periodic solution of (7). So we are reduced to the existence of a fixed point for the operator P_ε.

Now we introduce the auxiliary operator $P_\varepsilon^0 : C([0, T], K) \to C([0, T], \mathbb{R}^n)$, $P_\varepsilon^0 x(t) = x(T) + \varepsilon T f_0(x(T))$. Let us show that for $\varepsilon > 0$ sufficiently small the linear homotopy between P_ε and P_ε^0 on $\partial(C([0, T], K))$ is admissible. If we assume the contrary, then there exists sequences $\varepsilon_n \to 0$, $\lambda_n \in [0, 1]$, $\lambda_n \to \lambda_0$, $x_n \in \partial(C([0, T], K))$ such that

$$x_n(t) = \lambda_n \left(x_n(T) + \varepsilon_n \int_0^t f\big(s, x_n(s)\big)\,\mathrm{d}s \right) + (1 - \lambda_n)\big(x_n(T) + \varepsilon_n T f_0\big(x_n(T)\big)\big). \tag{15}$$

From this equality as $t = T$ it follows that

$$\lambda_n \left[\int_0^T f\big(s, x_n(s)\big)\,\mathrm{d}s - T f_0\big(x_n(T)\big) \right] + T f_0\big(x_n(T)\big) = 0. \tag{16}$$

Note, that since K is a compact set and $x_n(T) \in K$ then the sequence $x_n(T)$ is compact. Without of generality we can assume that $x_n(T)$ converges to some $x_0 \in \partial K$. From the boundedness of the set $C([0, T], K)$ and continuity of the function f it follows that the sequence x_n uniformly converges to x_0 as $n \to \infty$. Passing to the limit in (16) as $n \to \infty$ we obtain

$$f_0(x_0) = 0, \quad x_0 \in \partial K$$

which is a contradiction.

To conclude the proof we remark that Lemma 2.1 implies $\deg(-f_0, K) \neq 0$. Using the properties of the topological degree we have that for ε sufficiently small

$$\deg\big(I - P_\varepsilon, \partial\big(C\big([0, T], K\big)\big)\big) = \deg\big(I - P_\varepsilon^0, \partial\big(C\big([0, T], K\big)\big)\big) = \deg(-\varepsilon T f_0, K) = \deg(-f_0, K) \neq 0.$$

Thus, there exist fixed points of the operator P_ε and the set of T-periodic solutions of (7) is nonempty.

Let $\varepsilon_n \to 0$, x_{ε_n} be a corresponding sequence of T-periodic solutions of (7). The existence of the subsequence of x_{ε_n} which converges to some x_0, satisfying $f(x_0) = 0$ can be proved in the same way as we have proved the homotopy between P_ε and P_ε^0. For this in (15) we put $\lambda_n = 1$, $n \in N$. □

Now we formulate an existence of T-periodic solutions result for the system (9). We recall that $\bar{f}_0(\xi) = \eta(T, 0, \xi)$. We state now our second main result:

Theorem 3.2. *Let the functions $\phi, \psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable, K be an epilipschitz compact set. Suppose that the set $K^s(\bar{f}_0)$ is closed and the Euler characteristic $\chi(K^s(\bar{f}_0))$ is well defined. We also assume that $\bar{f}_0(x) \neq 0$ for every $x \in \partial K$ and that $\chi(K) \neq \chi(K^s(\bar{f}_0))$. Assume, that the following conditions hold*:

(A0) $\quad \Omega(T, 0, \xi) = \xi, \quad \xi \in \partial K,$

(A1) $\quad \eta(T, s, \xi) - \eta(0, s, \xi) \neq 0, \quad s \in [0, T],\ \xi \in \partial K.$

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$ system (9) *has at least one T-periodic solution x_ε such that $\Omega(0, t, x_\varepsilon(t)) \in K$, $t \in [0, T]$.*

By Lemma 2.1, $\deg(\eta(T, 0, \cdot), K) \neq 0$. The rest of the proof could be deduced from the following technical lemma:

Lemma 3.3. *(See [7, Theorem 1].) Let the set $U \subset \mathbb{R}^n$ be open and bounded, $K = \bar{U}$. Assume, that the conditions* (A0)*,* (A1)*, of Theorem* 3.2 *are satisfied. Suppose that*

(A2) $\quad \deg\big(\eta(T, 0, \cdot), K\big) \neq 0.$

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$ system (9) *has at least one T-periodic solution x_ε such that $\Omega(0, t, x_\varepsilon(t)) \in K$, $t \in [0, T]$.*

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