Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 347 (2009) 357–362
COMPTES RENDUS MATHEMATIQUE

Mathematical Analysis

Expanders and dimensional expansion

Jean Bourgain

Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA

Received and accepted 12 January 2009

Available online 6 March 2009

Presented by Jean Bourgain

Abstract

We construct finite families of $\mathrm{SL}_2(\mathbb{R})$ elements that are arbitrary close to identity and such that the corresponding Hecke operator, acting by Moebius transformation, has a uniform spectral gap (in a suitably restricted sense). This provides finite systems of monotone transformations of the interval [0, 1] with the expansion property. Combined with the approach from Dvir and Shpilka (2008), we obtain a solution to the "dimension expander" problem from Wigderson (2004). ***To cite this article: J. Bourgain, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Expanseurs et expansion dimensionnelle. On construit une famille finie d'éléments de $\mathrm{SL}_2(\mathbb{R})$, arbitrairement proches de l'identité, telle que l'opérateur de Hecke associé agissant par transformation de Moebius ait un trou spectral uniforme (en un sense restreint approprié).

Cela donne des systèmes finis de transformations monotones de l'intervalle ayant la propriété d'expansion. Ensuite, par l'approche de Dvir et Shpilka (2008), on obtient une solution au problème de Wigderson (2004) sur "l'expansion dimensionnelle". ***Pour citer cet article : J. Bourgain, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

On construit des familles de transformations linéaires $T_1, \dots, T_k$ de $\mathbb{F}^n$ ($\mathbb{F}$ un corps arbitraire et $n \to \infty$) telles que pour tout sous-espace W de $\mathbb{F}^n$, $\dim W \leqslant \frac{n}{2}$, on ait

$$\dim\left(W + \sum_{i=1}^{k} T_i W\right) \geqslant (1+\varepsilon)\dim W$$

où $\varepsilon > 0$ et une constante et k borné (indépendamment de n).

Le point de départ est l'approche de [3] ramenant la question à celle de produire un système de transformations monotones de $\{1, \dots, n\}$ ayant une propriété d'expansion. Pour $g \in \mathrm{SL}_2(\mathbb{R})$, soit $\bar{g}(x) = \frac{ax+b}{cx+d}$ la transformation de

E-mail address: bourgain@math.ias.edu.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.02.009

Moebius et $\rho_g f = (g')^{\frac{1}{2}}(f \circ \bar{g})$ la représentation unitaire correspondante. On résout le problème en montrant que pour toute $\varepsilon > 0$, il existe un sous-ensemble fini $\mathcal{G}$ de $\mathrm{SL}_2(\mathbb{R})$, tel que $\|1 - g\| < \varepsilon$ pour $g \in \mathcal{G}$ et l'opérateur de Hecke associé $T = \frac{1}{2|\mathcal{G}|}\sum_{g\in\mathcal{G}}(\rho_g + \rho_{g-1})$ vérifie

$$\|Tf\|_2 \leqslant \frac{1}{2}\|f\|_2$$

si $f \in L^2(\mathbb{R})$, supp $f \subset [0,1]$ et tel que $\int_{k-1/K}^{k/K} f(x)\,\mathrm{d}x = 0$ pour $1 \leqslant k \leqslant K$, où $K \in \mathbb{Z}_+$ depend de ε. La méthode est voisine de celle de [1] pour SU(2). L'étape finale (due à [5]) est effectuée ici par un argument plus général ayant d'autres applications.

1. Introduction and main statements

Let $\mathbb{F}$ be a field, $\varepsilon > 0$ and V a vector space of dimension n over $\mathbb{F}$. Following [8,4,3], a family $T_1, \dots, T_k$ of $\mathbb{F}$-linear transformations from V to V is called an ε-dimension expander if for every subspace W of V, $\dim W \leqslant \frac{n}{2}$,

$$\dim\left(W + \sum_{i=1}^{k} T_i W\right) \geqslant (1+\varepsilon)\dim W. \tag{1}$$

In [8] the problem was posed of producing explicit ε-dimension expanders of arbitrary large dimension. For fields of characteristic zero, the question was settled in [4] using property τ; their work left open the case of finite fields. Particularly relevant to this discussion is the contribution from [3] and we rely on their approach. Identifying V and F^n, their method is roughly as follows. Let $\varphi_1, \dots, \varphi_k$ be a family of monotone increasing transformations of $\{1, \dots, n\}$ i.e. $\varphi(x) \geqslant \varphi(y)$ if $x > y$. To each map φ, we associate a linear transformation T_φ of $\mathbb{F}^n$ defining

$$T_\varphi\left(\sum_{i=1}^{n} x_i e_i\right) = \sum x_i e_{\varphi(i)}. \tag{2}$$

As shown in [3], expander properties of $\{\varphi_1, \dots, \varphi_k\}$ imply dimension expansion for the resulting system $\{T_{\varphi_1}, \dots, T_{\varphi_k}\}$ of linear maps. Using shift maps, a dimension expander is produced in [3] involving $k \sim \log n$ linear maps. This is the best that may be achieved by shifts alone and different systems are needed to get k bounded independently of n. The missing ingredient is provided by the following.

Theorem 1. *There is $c_0 > 0$ and a explicit finite family Ψ of smooth increasing maps $\psi : [0,1] \to [0,1]$ such that for any measurable subset A of $[0,1]$, $|A| < \frac{1}{2}$,*

$$\max_{\psi\in\Psi}\bigl|\psi(A)\backslash A\bigr| \geqslant c_0|A|. \tag{3}$$

We proceed then as follows. To each $\psi \in \Psi$ we associate a transformation φ of $\{1, \dots, n\}$ defining $\varphi(i) = [n\psi(\frac{i}{n})]$ for $1 \leqslant i \leqslant n$ ($[x]$ denoting the integer part of $x \in \mathbb{R}_+$). Clearly φ is monotonically increasing. We add to this family the 1-shift φ_0

$$\begin{cases} \varphi_0(i) = i + 1 & \text{if } 1 \leqslant i < n, \\ \varphi_0(n) = n. \end{cases}$$

Denote Φ the resulting system of maps. Let then $D \subset \{1, \dots, n\}$, $|D| < \frac{n}{2}$ and let $D' = D \cup \bigcup_{\varphi\in\Phi}\varphi(D)$. Following [3], we need to show that

$$|D'| \geqslant (1+\varepsilon)|D| \tag{4}$$

for some $\varepsilon > 0$. Suppose (4) fails. Since by construction $\psi(\frac{i}{n}) \in \frac{1}{n}\varphi(i) + [0, \frac{1}{n}]$, we have $\psi(\frac{D}{n}) \subset \frac{1}{n}D' + [0, \frac{1}{n}]$ and $\psi(A) \subset \frac{1}{n}D' + [-\frac{c}{n}, \frac{c}{n}]$ with $A = \frac{1}{n}D + [0, \frac{1}{n}]$ and $c = 1 + \max_{\psi\in\Psi}|\psi'|_\infty$. Therefore

$$\bigl|\psi(A)\backslash A\bigr| \leqslant \frac{1}{n}\bigl(D + [-c,c]\bigr)\backslash\bigl(D + [0,1]\bigr) + \frac{1}{n}(2c+1)|D'\backslash D|$$
$$\frac{1}{n}(2c+1)\bigl(\bigl|\varphi_0(D)\backslash D\bigr| + |D'\backslash D|\bigr) + 0\left(\frac{1}{n}\right) < 0(\varepsilon)|A|$$

which is in violation of (3).

Theorem 1 will be derived from a spectral gap property for certain elements of $\mathrm{SL}_2(\mathbb{R})$. We believe this result is new and of independent interest.

Recall the action of an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R} \cup \{\infty\}$ by Moebius transformation $\bar{g}(x) = \frac{ax+b}{cx+d}$. Since $(\bar{g})'(x) = \frac{1}{(cx+d)^2} > 0$, $\bar{g}$ is increasing on any interval not containing $-\frac{d}{c}$.

Theorem 1 is easily derived from

Theorem 2. *For any $\varepsilon > 0$, there is a finite subset $\mathcal{G}$ of $\mathrm{SL}_2(\mathbb{R})$ and $K \in \mathbb{Z}_+$ with the following properties*

$$\|1 - g\| < \varepsilon \quad \text{for } g \in \mathcal{G} \tag{5}$$

and

$$\max_{g \in \mathcal{G}} \|f - f \circ \bar{g}\|_2 > \frac{1}{2} \tag{6}$$

for any function $f \in L^2(\mathbb{R})$, $\operatorname{supp} f \subset [0,1]$ such that $\|f\|_2 = 1$ and

$$\int_{(k-1)/K}^{k/K} f(x)\,\mathrm{d}x = 0 \quad \text{for all } 1 \leqslant k \leqslant K.$$

Let φ be the unitary representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{R})$ defined by

$$\varphi_{g^{-1}} f = (\bar{g}')^{1/2} (f \circ \bar{g}). \tag{7}$$

Since $\|1 - (\bar{g})'\|_\infty = 0(\varepsilon)$ on bounded sets, (6) will follow from

$$\left\langle \sum_g \nu(g) \rho_g f, f \right\rangle < \frac{1}{2} \tag{8}$$

with $\nu = \frac{1}{2|\mathcal{G}|} \sum_{g \in \mathcal{G}} (\delta_g + \delta_{g^{-1}})$ the symmetric probability measure on $\mathrm{SL}_2(\mathbb{R})$.

Statement (8) for functions f satisfying the condition in (6) is what we referred to as a "restricted" spectral gap property. The proof of (8) for a suitable family $\mathcal{G}$ introduced in the next section will rely on similar arguments as in [1] in the SU(2)-context and not on hyperbolicity.

2. Construction of certain free elements in $\mathrm{SL}_2(\mathbb{R})$

The set $\mathcal{G}$ in Theorem 2 is provided by

Lemma 1. *Given $\varepsilon > 0$, there is $Q \in \mathbb{Z}_+$ and $\mathcal{G} \subset \mathrm{SL}_2(\mathbb{R}) \cap (\frac{1}{Q} \mathrm{Mat}_2(\mathbb{Z}))$ satisfying*

$$\frac{1}{\varepsilon} < Q < \left(\frac{1}{\varepsilon}\right)^{c_1}, \tag{9}$$

$$|\mathcal{G}| > Q^{c_2}, \tag{10}$$

$$\textit{the elements of } \mathcal{G} \textit{ are free generators of a free group}, \tag{11}$$

$$\|g - 1\| < \varepsilon \quad \text{for } g \in \mathcal{G}. \tag{12}$$

Sketch of proof. A way to proceed is as follows. Start with the elements $g_1 = \begin{pmatrix} 1 & \frac{1}{q} \\ 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & 0 \\ \frac{1}{q} & 1 \end{pmatrix}$ and apply Breuillard's result [2] on the uniform Tits alternative. Since $\langle g_1, g_2 \rangle$ contains the free group $\langle g_1^q, g_2^q \rangle$, there are words $h_1, h_2 \in W_r(g_1, g_2)$, $r \in \mathbb{Z}$ an absolute constant, generating a free group. Taking $\ell \sim \log \frac{1}{\varepsilon}$, let $\mathcal{F} \subset W_\ell(h_1, h_2)$ be a set of free elements such that $\log |\mathcal{F}| \sim \ell$. Note also that

$$\|g\|, \|g^{-1}\| \leqslant \left(1 + \frac{1}{q}\right)^{r\ell} = \beta \quad \text{for } g \in \mathcal{F}. \tag{13}$$

Covering the ball $B(1,\beta) \subset \mathrm{Mat}_2$ by balls of radius $\varepsilon\beta^{-1}$, we may find $g_0 \in \mathcal{F}$ such that $|\mathcal{F}'| = |\mathcal{F} \cap B(g_0, \varepsilon\beta^{-1})| > (\varepsilon\beta^{-2})^4|\mathcal{F}| > |\mathcal{F}|^{\frac{1}{2}}$ (take $q > q(r)$ and $\ell > C \log \frac{1}{\varepsilon}$). The family $\mathcal{G} = g_0^{-1}.(\mathcal{F}' \backslash \{g_0\})$ satisfies the required properties with $Q = q^{2r\ell}$. $\square$

Since $W_k(\mathcal{G}) \subset Q^{-k}\,\mathrm{Mat}_2(\mathbb{Z})$, distinct elements of $W_k(\mathcal{G})$ are at least Q^{-k}-apart. From Kesten's bound in the free group $F_{\mathcal{G}}$, we have

$$\max_g \nu^{(k)}(g) \lesssim \left(\frac{\sqrt{2|\mathcal{G}|-1}}{|\mathcal{G}|}\right)^k \tag{14}$$

for the k-fold convolution $\nu^{(k)}$ of ν introduced above.

3. Proof of Theorem 2 (I)

This part of the argument is closely related to [1]. Recall that

$$\nu = \frac{1}{2|\mathcal{G}|}\sum_{g\in\mathcal{G}}(\delta_g + \delta_{g^{-1}}).$$

Let P_δ, $\delta > 0$ be an approximate identity on $\mathrm{SL}_2(\mathbb{R})$. The main statement is the following

Lemma 2. *Given $\tau > 0$, we have*

$$\left\|\nu^{(\ell)} * P_\delta\right\|_\infty < \delta^{-\tau} \tag{15}$$

provided

$$\ell > C_3(\tau)\frac{\log(1/\delta)}{\log Q} \tag{16}$$

and assuming δ small enough (depending on Q and τ).

Here $\| \ \|_\infty$ stands for the L^∞-norm on $\mathrm{SL}_2(\mathbb{R})$.

The proof of Lemma 2 is similar to its analogue for SU(2) (cf. [1]) and relies on techniques from arithmetic combinatorics. Note that the analogy is not surprising as $\mathrm{SL}_2(\mathbb{R})$ and SU(2) have the same Lie-algebra, up to complexification.

Start by taking ℓ_0 s.t. $Q^{-\ell_0} > \delta > Q^{-2\ell_0}$. Then $\mu = \nu^{(\ell_0)} * P_\delta$ satisfies by (14), (10) $\|\mu\|_\infty \lesssim \delta^{-3}|\mathcal{G}|^{-\frac{1}{2}\ell_0} < \delta^{-3+c_4}$. One shows then that for $r > r(\tau)$, the convolution $\mu^{(r)}$ satisfies (15). The key ingredient is the "L^2-flattening" principle (see [1], Prop. 2.2). Let $\mu = \nu^{(\ell)} * P_\delta$, $\ell_0 \leqslant \ell \lesssim \ell_0$ satisfy for some $\gamma > 0$

$$\delta^{-\gamma} < \|\mu\|_2 < \delta^{-\frac{3}{2}+\gamma}. \tag{17}$$

Then for some $\sigma = \sigma(\gamma) > 0$

$$\|\mu * \mu\|_2 < \delta^{\sigma}\|\mu\|_2. \tag{18}$$

We briefly review the different steps of the proof with reference to [1].

(i) Assume (18) fails. We apply the Balog–Szemeredi–Gowers theorem in $\mathrm{SL}_2(\mathbb{R})$, which is a "locally reasonable metric group" (in the sense of [6,7]). Application of [6], Theorem 6.10, with $K = \mathrm{SL}_2(\mathbb{R}) \cap B\big(1, (1+\varepsilon)^\ell\big)$ and $K = \delta^{-o(1)}$ gives an "approximative group" $H \subset \mathrm{SL}_2(\mathbb{R})$ such that $\mu(xH) > \delta^{-o(1)}$ for some $x \in \mathrm{SL}_2(\mathbb{R})$.
(ii) Construction of a large set of traces (see [1], Lemma 5.2).
(iii) Construction of a large set of simultaneously diagonalizable elements ([1], Lemma 5.1).
(iv) Amplification of the trace set. See [1], Lemma 5.4 based on the "discretized ring theorem" (which is the underlying scalar sum-product amplification).
(v) From [1], Corollary 5.2, it follows that the metric entropy $\mathcal{N}(H^{(s)}, \delta) > \delta^{-c}\mathcal{N}(H, \delta)$ for some $s \in \mathbb{Z}_+$ and $c = c(\gamma) > 0$. But this contradicts the fact that H is an approximative group.

Note that in what follows we will apply Lemma 2 for a specific, sufficiently small value of τ.

4. Proof of Theorem 2 (II)

The final step in proving the expansion for SU(2) in [1] is an argument due to Sarnak and Xue from representation theory (see [5]). What follows is an alternative and quite general approach (see also Section 5).

Let f be as in (6) and assume (8) fails. Let $f = \sum_{k=0}^{\infty} \Delta_k f$ be Littlewood–Paley decomposition, i.e. $|\lambda| \sim 2^k$ for $\lambda \in \operatorname{supp} \widehat{\Delta_k f}$, $k \geqslant 1$. Denote the Hecke operator

$$T = \sum \nu(g)\rho_g. \tag{19}$$

It follows that

$$\|T(F)\|_2 > \frac{1}{20} \tag{20}$$

with $F = \frac{\Delta_k f}{\|\Delta_k f\|_2}$ for some $k \geqslant k_0$. Note that k_0 can be made arbitrarily large by suitable choice of K in (6). Let $\delta = 4^{-k}$ and $k < \ell \log Q < ck$ such that $\|\mu = \nu^{(\ell)} * P_\delta\|_\infty < \delta^{-\tau}$, $\tau = 10^{-2}$, according to Lemma 2. From (20)

$$\left\| \int \rho_g F \mu(\mathrm{d}g) \right\|_2 > 30^{-\ell} \tag{21}$$

implying

$$10^{-3\ell} < \delta^{-2\tau} \int\limits_{\Omega\times\Omega} \left|\langle \rho_g F, \rho_h F\rangle\right| \mathrm{d}g\,\mathrm{d}h < \delta^{-2\tau}(1+\varepsilon)^{c\ell} \int\limits_{\Omega.\Omega} \left|\langle F, \rho_g F\rangle\right| \mathrm{d}g \tag{22}$$

with $\Omega = B(1, (1+\varepsilon)^\ell)$.

We decompose $F = F_1 + F_\infty$ with $\|F_1\|_1 \leqslant 2^{-\alpha k}$, $\|F_\infty\|_\infty \leqslant 2^{\alpha k}$, $\alpha = \frac{1}{20}$ and F_i $(i = 1, \infty)$ satisfying the same Fourier transformation restriction.

Parametrizing $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} u\cos\theta & v\cos\psi \\ u\sin\theta & v\sin\psi \end{smallmatrix}\right)$, $uv\sin(\psi-\theta) = 1$, we have $\mathrm{d}g = \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{a} = \frac{\mathrm{d}u\,\mathrm{d}\theta\,\mathrm{d}\psi}{u\sin^2(\theta-\psi)}$ on the chart $a \neq 0$. Thus $|\mathrm{d}g| < \|g\|^3\,\mathrm{d}u\,\mathrm{d}\theta\,\mathrm{d}\psi$.

Estimate (22) for $F = F_i$. For $i = 1$, (22) $\leqslant \delta^{-2\tau}(1+\varepsilon)^{c\ell}\|F\|_1 . \int |F_1(\operatorname{cotg} g\psi)|\,\mathrm{d}\psi < (1+\varepsilon)^{c\ell}\delta^{-2\tau}4^{-\alpha k}$. Let ψ be a smooth function on $\mathrm{SL}_2(\mathbb{R})$, $\psi(g) = 1$ for $\|g\| < (1+\varepsilon)^{2\ell}$, $\psi(g) = 0$ for $\|g\| > 2(1+\varepsilon)^{2\ell}$. For $F = F_\infty$

$$(22)^2 < \delta^{-4\tau}(1+\varepsilon)^{c\ell}4^{\alpha k} \int_0^1\!\!\int_0^1 \left| \int \rho_g F(x)\rho_g \bar{F}(y)\psi(g)\,\mathrm{d}g \right| \mathrm{d}x\,\mathrm{d}y < (1+\varepsilon)^{c\ell}\delta^{-4\tau}4^{(2\alpha-\frac{1}{6})k}$$

(here we used that $\operatorname{supp}\hat{F} \subset [|\lambda| \sim 2^k]$).

Collecting these estimates gives (21) $< (1+\varepsilon)^{c\ell}(4^{k(\tau-\frac{\alpha}{2})} + 2^{\frac{1}{2}k(\tau+2\alpha-\frac{1}{6})}) < 2^{-\frac{k}{50}}$, contradicting $\ell.\log Q < ck$ if Q is large enough.

Remark. Previous argument is clearly of a general nature and applies in other situations. In fact it becomes more transparent in the case of a finite or compact group. Consider for instance the representation of $G = \mathrm{SO}(d)$ on $L^2(S_{d-1})$, $\rho_g f = f \circ g$, for $d \geqslant 3$.

Denote again $f = \sum \Delta_k f$ a Littlewood–Paley decomposition for $f \in L^2(S_{d-1})$ (which may be realized using spherical harmonics). Taking F as above, we obtain in (22)

$$\delta^{-2\tau} \int\limits_G \left|\langle F, \rho_g F\rangle\right| \mathrm{d}g. \tag{23}$$

Clearly

$$(23)^2 \leqslant \delta^{-4\tau} \int\limits_{S_{d-1}^3} \left|F(x)\right|\left|F(y)\right|\left|F(z)\right|\left|(A_{\theta(x,y)}F)(z)\right| \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{24}$$

with $\theta(x, y)$ the angle between $x, y \in S$ and where $A_\theta F(z)$ denotes the average of F on the $(d-2)$-sphere $[\zeta \in S_{d-1}; \theta(z, \zeta) = \theta(x, y)]$. Since $F = \Delta_k F$, standard bounds on spherical averaging operators imply $\|A_\theta F\|_{L^2(S_{d-1})} \leqslant C(1 + 2^k|\sin\theta|)^{-\frac{d}{2}+1}$ and hence (24) $< \delta^{-4\tau} 2^{-k\frac{d-2}{2}}$.

References

[1] J. Bourgain, A. Gamburd, On the spectral gap for finitely-generated subgroups of SU(2), Invent. Math. 171 (1) (2008) 83–121.
[2] E. Breuillard, A strong Tits alternative, preprint 08.
[3] Z. Dvir, A. Shpilka, Towards dimension expanders over finite fields, preprint 08.
[4] A. Lubotzky, Y. Zelmanov, Dimension expanders, preprint 08.
[5] P. Sarnak, X. Xue, Bounds for multiplicities of automorphic representations, Duke Math J. 64 (1991) 207–227.
[6] T. Tao, Product sets estimates for non-commutative groups, Combinatorica, in press.
[7] T. Tao, V. Vu, Additive Combinatorics, Cambridge University Press, 2006.
[8] A. Wigderson, A lecture at IPAM, UCLA, February 2004.