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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 243–247

COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Scattering for small energy solutions of NLS with periodic potential in 1D

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Received 21 August 2008; accepted after revision 27 January 2009

Available online 20 February 2009

Presented by Jean-Michel Bony

Abstract

Given $H \equiv -\partial_x^2 + V(x)$ with $V : \mathbb{R} \to \mathbb{R}$ a smooth periodic potential, for $\mu \in \mathbb{R}\backslash\{0\}$ and $p \geqslant 7$, we prove scattering for small solutions to

$$\mathrm{i}\partial_t u + Hu = \mu|u|^{p-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \qquad u(0) = u_0 \in H^1(\mathbb{R}).$$

To cite this article: S. Cuccagna, N. Visciglia, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

Résumé

Scattering pour NLS avec des données petites et potentiel périodique. On étudie l'existence de l'opérateur de scattering pour le problème de Cauchy suivant :

$$\mathrm{i}\partial_t u + Hu = \mu|u|^{p-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \qquad u(0) = u_0 \in H^1(\mathbb{R})$$

où $H \equiv -\partial_x^2 + V(x)$, $V : \mathbb{R} \to \mathbb{R}$ est un potentiel périodique régulier, $\mu \in \mathbb{R}\backslash\{0\}$, $p \geqslant 7$ et u_0 est une petite donnée dans l'espace $H^1(\mathbb{R})$. ***Pour citer cet article : S. Cuccagna, N. Visciglia, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Étant donné un potentiel périodique et régulier $V : \mathbb{R} \to \mathbb{R}$ on considère le problème de Cauchy suivant :

$$\mathrm{i}\partial_t u - \partial_x^2 u + V(x)u = \mu|u|^{p-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \qquad u(0) = u_0 \in H^1(\mathbb{R}) \tag{1}$$

avec des données initiales $u_0(x)$ petites en $H^1(\mathbb{R})$. Depuis l'article [2] il est bien connu qu' on a l'éstimation de dispersion suivante :

$$\left\|\mathrm{e}^{\mathrm{i}t(-\partial_x^2+V)} f\right\|_{L^\infty(\mathbb{R})} \leqslant C\,\mathrm{Max}\left\{|t|^{-\frac{1}{2}}, \langle t\rangle^{-\frac{1}{3}}\right\}\|f\|_{L^1(\mathbb{R})}. \tag{2}$$

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.01.028

Donc, au moins localement en temps, on a la même estimation de dispersion que l'ont a pour le propagateur libre $e^{-it\partial_x^2}$. Cela implique, à l'aide de l'argument abstrait du TT^* (voir [5]), que les éstimations classiques de Strichartz sont satisfaites localement en temps par le propagateur $e^{it(-\partial_x^2+V)}$. En vertu de ce fait on a que le problème de Cauchy (1) est bien posé soit localement soit globalement en temps, pourvu que les données initiales sont petites (voir [1]).

Donc le point nouveau de cet article repose dans la démonstration de l'existence de l'opérateur de scattering. La difficulté principale est liée au fait que la dispersion donnée par (2) est plus faible par rapport a celle que on a dans le cas libre $e^{-it\partial_x^2}$, si on considére des grands temps.

À l'aide des éspaces de Birman–Solomjak (voir (4)) on va écrire des éstimations de Strichartz appropriées pour le propagateur $e^{it(-\partial_x^2+V)}$. En outre il sera nécessaire de couper le propagateur $e^{it(-\partial_x^2+V)}$ à l'aide d'un projecteur π qu'on introduit explicitement dans le Lemme 2.2 et qui va nous permettre d'écrire $e^{it(-\partial_x^2+V)}$ comme la somme de deux propagateurs pour lesquelles on pourra écrire les correspondantes éstimations de Strichartz. Finalement on va appliqer les éstimations de Strichartz qu'on trouve pour démontrer que si $p \geqslant 7$ et si $u_0 \in H^1(\mathbb{R})$ est assez petit alors il est bien défini l'opérateur de scattering, i.e. :

$$\exists \epsilon, C > 0 \text{ tels que } \forall u_0 \in H^1(\mathbb{R}) \text{ avec } \|u_0\|_{H^1(\mathbb{R})} < \epsilon$$
$$\exists u_\pm \in H^1(\mathbb{R}) \text{ avec } \|u_\pm\|_{H^1(\mathbb{R})} < C\|u_0\|_{H^1(\mathbb{R})} \text{ et}$$
$$\lim_{t\to\pm\infty} \left\| u(t,x) - e^{it(-\partial_x^2+V)}u_\pm \right\|_{H^1(\mathbb{R})} = 0$$

où $u(t,x)$ dénote la solution de (1).

1. Introduction

In this Note, for $\beta : \mathbb{R}^+ \to \mathbb{R}$ a suitable nonlinearity, we prove scattering of small solutions of

$$i\partial_t u + Hu = \beta(|u|^2)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \qquad u(0) = u_0 \in H^1(\mathbb{R}) \tag{3}$$

where $H \equiv -\partial_x^2 + V(x)$ with $V(x)$ a smooth real valued periodic potential. To do this we need to write appropriate Strichartz estimates for H. For every $1 \leqslant p, q \leqslant \infty$ we consider the Birman–Solomjak spaces

$$\ell^p(\mathbb{Z}, L_t^q[n,n+1]) \equiv \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}) \text{ s.t. } \left\{ \|f\|_{L^q[n,n+1]} \right\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}) \right\}, \tag{4}$$

endowed with the natural norms

$$\|f\|^p_{\ell^p(\mathbb{Z},L_t^q[n,n+1])} \equiv \sum_{n\in\mathbb{Z}} \|f\|^p_{L_t^q[n,n+1]} \quad \forall\, 1 \leqslant p < \infty \text{ and } 1 \leqslant q \leqslant \infty,$$
$$\|f\|_{\ell^\infty(\mathbb{Z},L_t^q[n,n+1])} \equiv \sup_{n\in\mathbb{Z}} \|f\|_{L^q[n,n+1]}.$$

We consider the Sobolev spaces

$$W^{k,p}(\mathbb{R}) \equiv \left\{ f \in \mathcal{S}'(\mathbb{R}) \mid \left(1-\partial_x^2\right)^{k/2} f \in L^p(\mathbb{R}) \right\}. \tag{5}$$

For $p = 2$ we set $H^k(\mathbb{R}) \equiv W^{k,2}(\mathbb{R})$. Then we prove:

Theorem 1.1. *Assume $\beta(t) \in C^3(\mathbb{R}, \mathbb{R}^3)$ with $\beta(0) = \beta'(0) = \beta''(0) = 0$ and that $V(x)$ is a smooth periodic and nonconstant real valued potential. Then there exists $\epsilon_0 > 0$ such that for any initial data $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|_{H^1(\mathbb{R})} < \epsilon_0$ problem (3) is globally well-posed. Moreover there exists $C = C(\epsilon_0) > 0$ such that it is possible to split $u(t,x) = u_1(t,x) + u_2(t,x)$ so that for any couple (r,p) that satisfies*

$$2/r + 1/p = 1/2 \quad \text{and} \quad (r,p) \in [4,\infty] \times [2,\infty], \tag{6}$$

we have

$$\left\| u_1(t,x) \right\|_{\ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n,n+1], W^{1,p}(\mathbb{R})))} + \left\| u_2(t,x) \right\|_{L_t^r(\mathbb{R}, W^{1,p}(\mathbb{R}))} \leqslant C\|u_0\|_{H^1(\mathbb{R})}. \tag{7}$$

Furthermore, there exist $u_\pm \in H^1(\mathbb{R})$ with $\|u_\pm\|_{H^1(\mathbb{R})} < C\|u_0\|_{H^1(\mathbb{R})}$ such that

$$\lim_{t\to\pm\infty} \left\| u(t,x) - e^{-itH}u_\pm \right\|_{H^1(\mathbb{R})} = 0. \tag{8}$$

If $V(x)$ is constant there is a considerable literature on (3). A basic tool are the Strichartz estimates, see [1,5], which follow, for $\mathcal{V}(t) \equiv e^{it\partial_x^2}$, from

$$\|\mathcal{V}(t)f\|_{L^\infty(\mathbb{R})} \leqslant C|t|^{-\frac{1}{2}}\|f\|_{L^1(\mathbb{R})}. \tag{9}$$

For any $V(x)$ not constant (9) is not true and by [2] we have instead

$$\|e^{itH}f\|_{L^\infty(\mathbb{R})} \leqslant C\,\mathrm{Max}\{|t|^{-\frac{1}{2}}, \langle t\rangle^{-\frac{1}{3}}\}\|f\|_{L^1(\mathbb{R})}. \tag{10}$$

(10) requires a new set of Strichartz estimates for e^{itH}. This is done in Section 2 in the context of the Birman–Solomjak spaces. We remark that $\|w\|_{L_t^{3r/2}(\mathbb{R},W^{1,p}(\mathbb{R}))} \leqslant \|w\|_{\ell^{3r/2}(\mathbb{Z},L_t^\infty([n,n+1],W^{1,p}(\mathbb{R})))}$ and that Strichartz estimates with $L_t^{3r/2}(\mathbb{R}, W^{1,p}(\mathbb{R}))$ spaces are sufficient to prove (8) (see Remark 3.1). Since Strichartz estimates with Birman–Solomjak can be proved with the same amount of effort required for $L_t^r(\mathbb{R}, W^{1,p}(\mathbb{R}))$ spaces, see [3], and yield stronger results, we chose to use them, even if they are less standard.

In Section 3 we apply the Strichartz estimates to the nonlinear problem.

In the sequel we shall use the following notations: $L_x^p = L^p(\mathbb{R})$, $W_x^{k,p} = W^{k,p}(\mathbb{R})$, $H_x^s = H^s(\mathbb{R})$, for any $r \in [1,\infty]$ we set $r' = \frac{r}{r-1}$.

2. Strichartz estimates

By standard arguments, see [3] (Lemma 3.2 and its proof in §9) and [4], it is possible to prove:

Lemma 2.1. *Let $\mathcal{U}(t): L_x^2 \to L_x^2$ be a uniformly bounded group in L_x^2 such that $\|\mathcal{U}(t)f\|_{L_x^\infty} \leqslant C_1\langle t\rangle^{-\frac{1}{3}}\|f\|_{L_x^1}$. Let $I \ni 0$ be an interval and let $\chi_I(t)$ its characteristic function. Then there exists $c_0 = c_0(C_1)$, independent of I, such that, for every pair which satisfies* (6)*, we have*

$$\|\chi_I(t)\mathcal{U}(t)f\|_{\ell^{\frac{3}{2}r}(\mathbb{Z},L_t^\infty([n,n+1],L_x^p))} \leqslant c_0\|f\|_{L_x^2} \tag{11}$$

and such that, for any two pairs (r_1, p_1) and (r_2, p_2) that satisfy (6)*, we have*

$$\left\|\chi_I(t)\int_0^t \mathcal{U}(t-s)F(s)\,ds\right\|_{\ell^{\frac{3}{2}r_1}(\mathbb{Z},L_t^\infty([n,n+1],L_x^{p_1}))} \leqslant c_0\|\chi_I F\|_{\ell^{(\frac{3}{2}r_2)'}(\mathbb{Z},L_t^1([n,n+1],L_x^{p_2'}))}. \tag{12}$$

Our next step is:

Lemma 2.2. *There exists a projection $\pi: L_x^2 \to L_x^2$ which commutes with e^{itH} such that the group $\mathcal{U}(t) \equiv \pi e^{itH}$ satisfies the hypotheses of Lemma* 2.1 *and the group $\mathcal{V}(t) \equiv (1-\pi)e^{itH}$ satisfies the estimate* (9).

Proof. We have $e^{itH}(x,y) = K(t,x,y)$

$$K(t,x,y) = \int_{\mathbb{B}} e^{i(tE(k)-(x-y)k)} m_-^0(x,k)m_+^0(y,k)\,dk$$

with $e^{\mp ixk}m_\mp^0(x,k)$ the Bloch functions and $E(k)$ the band function, see [2]. By §4 [2] there are two characteristic functions $\chi_j(k)$, $j = 1,2$, such that $1 = \chi_1(k) + \chi_2(k)$ in $\mathbb{R}$ and such that, if we set

$$K_j(t,x,y) = \int_{\mathbb{R}} e^{i(tE(k)-(x-y)k)} m_-^0(x,k)m_+^0(y,k)\chi_j(k)\,dk,$$

then there is a fixed $C > 0$ such that $|K_1(t,x,y)| \leqslant C\langle t\rangle^{-\frac{1}{3}}$ and $|K_2(t,x,y)| \leqslant C|t|^{-\frac{1}{2}}$ for all $(t,x,y) \in \mathbb{R}^3$. Notice that [2] treats the generic case when all the spectral gaps of the spectrum $\sigma(H)$ of H are nonempty, but the arguments are the same in the case $\sigma(H)$ has infinitely many bands with some empty gaps, and much easier if $\sigma(H)$ has finitely many bands. □

Notice that $\mathcal{V}(t)$ satisfies the standard Strichartz estimates. In the sequel we will pick for c_0 a constant for which all Strichartz estimates for both groups $\mathcal{U}(t)$ and $\mathcal{V}(t)$ are true.

3. Proof of Theorem 1.1

The global well posedness in H_x^1 is well know since it follows from standard theory. In particular for the local well posedness see Remark 3.5.4 [1]. The fact that we consider only small energy initial data yields global well posedness applying Corollary 3.5.3 [1] for $g(u) = V(x)u - \beta(|u|^2)u$. Summing up this yields:

(1) if $\|u_0\|_{H_x^1} < \epsilon \leqslant \epsilon_0$ with ϵ_0 sufficiently small, (3) admits a solution $u(t) \in L_t^\infty(\mathbb{R}, H_x^1) \cap W_t^{1,\infty}(\mathbb{R}, H_x^{-1})$;
(2) the above solution is unique;
(3) the above solution is $u(t) \in C^0(\mathbb{R}, H_x^1) \cap C^1(\mathbb{R}, H_x^{-1})$ and the following quantities are conserved:

$$\big\|u(t)\big\|_{L_x^2} = \|u_0\|_{L_x^2},$$
$$E(t) = \int_{\mathbb{R}} \left(\left|\partial_x u(t,x)\right|^2 - V(x)\left|u(t,x)\right|^2 + 2F\left(\left|u(t,x)\right|^2\right)\right) \mathrm{d}x = E(0)$$

where $F(0) = 0$ and $\partial_{\bar{u}} F(|u|^2) = \beta(|u|^2)u$;
(4) there exists a fixed $C > 0$ such that $\|u(t)\|_{H_x^1} < C\epsilon$ for all $t \in \mathbb{R}$.

Hence we need only to prove the scattering part. Recall that $u(t,x)$ satisfy the Duhamel's formula

$$u(t) = \mathrm{e}^{-\mathrm{i}tH} u_0 - \mathrm{i} \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)H} \beta\left(\left|u(s)\right|^2\right) u(s)\, \mathrm{d}s. \tag{13}$$

We will need the following lemma:

Lemma 3.1. *The following estimate holds*:

$$\left\|\beta\left(|u|^2\right)u\right\|_{L_t^1(\mathbb{R}, H_x^1)} \lesssim \|u_0\|_{H_x^1} \tag{14}$$

where $u(t,x)$ solves (3) *with β as in Theorem* 1.1 *and $\|u_0\|_{H_x^1} \leqslant \epsilon_0$ for a suitable $\epsilon_0 > 0$.*

Proof. By combining hypothesis on β with (4) above, we get over any bounded interval I

$$\begin{aligned}\left\|\beta\left(|u|^2\right)u\right\|_{L_t^1(I, H_x^1)} &\lesssim \left\||u|^6 u\right\|_{L_t^1(I, H_x^1)} \\ &\lesssim \|u\|_{L_t^\infty(I, H_x^1)} \|u\|_{L_t^6(I, L_x^\infty)}^6 \lesssim \|u_0\|_{H_x^1} \|u\|_{L_t^6(I, L_x^\infty)}^6.\end{aligned} \tag{15}$$

We split $u = u_1 + u_2$ with $u_1(t) = \pi u(t)$ and $u_2(t) = (1-\pi)u(t)$. By combining Lemma 2.1, Lemma 2.2 with the following identity:

$$u_1(t) = \pi \mathrm{e}^{-\mathrm{i}tH} u_0 + \pi \left(-\mathrm{i} \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)H} \beta\left(\left|u(s)\right|^2\right) u(s)\, \mathrm{d}s\right), \tag{16}$$

we get

$$\begin{aligned}\|u_1\|_{L_t^6(I, W_x^{1,\infty})} &\lesssim \|\chi_I u_1\|_{\ell^6(\mathbb{Z}, L_t^\infty[n,n+1]), W_x^{1,\infty}} \\ &\lesssim \|u_0\|_{H_x^1} + \left\|\beta\left(|u|^2\right)u\right\|_{L_t^1(I, H_x^1)} \lesssim \|u_0\|_{H_x^1} + \|u_0\|_{H_x^1} \|u\|_{L_t^6(I, L_x^\infty)}^6.\end{aligned} \tag{17}$$

By a similar argument we deduce the following estimate for u_2:

$$\|u_2\|_{L_t^6(I, W_x^{1,6})} \lesssim \|u_0\|_{H_x^1} + \|u_0\|_{H_x^1} \|u\|_{L_t^6(I, L_x^\infty)}^6. \tag{18}$$

By combining (17) and (18) with the Sobolev embedding $W^{1,6}(\mathbb{R}) \subset L^\infty(\mathbb{R})$ we deduce

$$\|u\|_{L_t^6(I,L_x^\infty)} \lesssim \|u_0\|_{H_x^1} + \|u_0\|_{H_x^1}\|u\|^6_{L_t^6(I,L_x^\infty)},$$

which implies for C independent of I

$$\|u\|_{L_t^6(I,L_x^\infty)} \leqslant C\|u_0\|_{H_x^1}$$

provided that u_0 is small in H_x^1. This estimate and (15) yield Lemma 3.1. □

Remark 3.1. We remark that the following inequality holds for (r, p) admissible:

$$\|w\|_{L_t^{3r/2}(\mathbb{R},W_x^{1,p})} \leqslant \|w\|_{\ell^{3r/2}(\mathbb{Z},L_t^\infty([n,n+1],W_x^{1,p}))}.$$

This inequality implies validity of Strichartz estimates in the spaces $L_t^{3r/2}W_x^{1,p}$ for the propagator πe^{itH}. In fact this weaker version of Strichartz estimates is sufficient to prove Lemma 3.1.

The proof of (7) follows in a straightforward way by combining the Strichartz estimates for the propagators πe^{itH} and $(1-\pi)e^{itH}$ (see Lemmas 2.1, 2.2) with (14). Finally we prove (8) via standard arguments. Notice that

$$e^{itH}u(t) = u_0 - i\int_0^t e^{isH}\beta\big(|u(s)|^2\big)u(s)\,ds$$

and so for $t_1 < t_2$

$$e^{it_2H}u(t_2) - e^{it_1H}u(t_1) = -i\int_{t_1}^{t_2} e^{isH}\beta\big(|u(s)|^2\big)u(s)\,ds.$$

Then by Lemma 3.1

$$\begin{aligned}\big\|e^{it_2H}u(t_2) - e^{it_1H}u(t_1)\big\|_{H_x^1} &\leqslant \left\|\int_{t_1}^{t_2} e^{isH}\beta\big(|u(s)|^2\big)u(s)\,ds\right\|_{H_x^1}\\ &\leqslant \big\|\beta\big(|u|^2\big)u\big\|_{L^1([t_1,t_2],H_x^1)} \to 0 \quad \text{for } t_1\to\infty \text{ and } t_1<t_2.\end{aligned} \tag{19}$$

Then $u_+ = \lim_{t\to\infty} e^{itH}u(t)$ satisfies the desired properties. One proves the existence of $u_- = \lim_{t\to-\infty} e^{itH}u(t)$ similarly.

References

[1] T. Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes in Mathematics, vol. 10, New York University, Courant Institute of Mathematical Sciences, New York, 2003.

[2] S. Cuccagna, Dispersion for Schrödinger equation with periodic potential in 1D, Comm. Partial Differential Equations 33 (11) (2008) 2064–2095.

[3] S. Cuccagna, N. Visciglia, On asymptotic stability of ground states of NLS with a finite bands periodic potential in 1D, arXiv:0809.4775.

[4] J. Ginibre, G. Velo, Time decay of finite energy solutions of the nonlinear Klein Gordon and Schrödinger equations, Ann. Inst. H. Poincaré A 43 (1985) 399–442.

[5] M. Keel, T. Tao, Endpoint Strichartz estimates, Amer. J. Math. 120 (5) (1998) 955–980.