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Partial Differential Equations

Finite time blow-up for radially symmetric solutions to a critical quasilinear Smoluchowski–Poisson system

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Abstract

Finite time blow-up is shown to occur for radially symmetric solutions to a critical quasilinear Smoluchowski–Poisson system provided that the mass of the initial condition exceeds an explicit threshold. In the supercritical case, blow-up is shown to take place for any positive mass. The proof relies on a novel identity of virial type. ***To cite this article: T. Cieślak, P. Laurençot, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Explosion en temps fini des solutions à symétrie radiale d'un système de Smoluchowski–Poisson quasilinéaire critique. L'explosion en temps fini est établie pour des solutions à symétrie radiale d'un système de Smoluchowski–Poisson quasilinéaire critique dès que la masse de la donnée initiale dépasse un certain seuil. Dans le cas surcritique, l'explosion peut se produire pour toute masse positive. L'argument principal de la démonstration est une nouvelle identité de type viriel. ***Pour citer cet article : T. Cieślak, P. Laurençot, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

L'objet de cette Note est l'étude de l'explosion en temps fini des solutions à symétrie radiale du système de Smoluchowski–Poisson généralisé (1)–(4), où $B(0,1)$ désigne la boule unité de $\mathbb{R}^n$, $n \geqslant 2$, et M est la moyenne (spatiale) de u_0. Le coefficient de diffusion $a \in \mathcal{C}^2([0,\infty))$ est supposé positif. Une propriété fondamentale des solutions de (1)–(4) est la positivité de u et l'invariance de sa moyenne (5). Lorsque $a \equiv 1$, il existe des solutions de (1)–(4) à symétrie radiale qui explosent en temps fini et ce phénomène se produit pour tout $M > 0$ si $n \geqslant 3$ mais seulement pour $M > 8\pi$ si $n \geqslant 2$ [5,7]. Plus récemment, il a été établi dans [4] que deux comportements radicalement différents sont observés en fonction de la diffusion : d'une part, si $a(z) \geqslant c(1+z)^\alpha$ et $\alpha > (n-2)/n$, les solutions de

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(1)–(4) sont globales. D'autre part, si $a(z) \leqslant c\ (1+z)^\alpha$ et $\alpha < (n-2)/n$, des solutions de (1)–(4) à symétrie radiale explosant en temps fini sont construites dans [4].

Le cas critique $\alpha = (n-2)/n$ n'est pas inclus dans les travaux précédemment mentionnés si $n \geqslant 3$ et l'objet de cette Note est de montrer qu'il y a aussi des solutions de (1)–(4) à symétrie radiale explosant en temps fini dans ce cas dès que M est suffisamment grand. Nous montrons de plus que, dans le cas surcritique $\alpha < (n-2)/n$, ce phénomène d'explosion peut se produire pour tout $M > 0$. Notre approche est différente de celle utilisée dans [4,5] et utilise une inégalité différentielle qui se révèle être incompatible avec l'existence globale. Lorsque $a \equiv 1$, cette technique est employée dans [7] : le moment d'ordre n de u vérifie une inégalité différentielle qui est en fait incompatible avec la positivité (5) de u pour des grands temps. Dans le cas d'une diffusion non linéaire, cette inégalité ne semble pas permettre de conclure et nous développons de nouvelles identités de ce type (cf. Lemma 2.2) pour démontrer le résultat suivant :

Théorème 0.1. *Supposons qu'il existe $\alpha \in [0, (n-2)/n]$, $c_1 > 0$ et $c_2 > 0$ telle que $0 < a(z) \leqslant c_1\ z^\alpha + c_2$, $z \geqslant 0$. Soit $M > 0$ et $u_0 \in L^\infty(B(0,1))$ une fonction positive à symétrie radiale telle que $\|u_0\|_1 = M$. S'il existe $p > 1$ telle que $E_{M,p}(\bar{m}_p(u_0)) < 0$, les fonctions $\bar{m}_p$ et $E_{M,p}$ étant définies par (7) et (9), alors le système (1)–(4) a une unique solution classique (u, v) dont le temps maximal d'existence $T_{\max} \in (0,\infty)$ est fini et telle que $\|u(t)\|_\infty \to \infty$ lorsque $t \to T_{\max}$.*

Il existe effectivement des conditions initiales u_0 pour lesquelles $E_{M,p}(\bar{m}_p(u_0)) < 0$ quelle que soit la valeur de $M > 0$ si $\alpha \in [0, (n-2)/n)$ ou pour M est assez grand si $\alpha = (n-2)/n$. En effet, il suffit de choisir une fonction u_0 dont le support est inclus dans $B(0,\delta)$ telle que $u_0 = M\delta^{-n}\ \mathbf{1}_{B(0,\delta)}$ pour $\delta > 0$ suffisamment petit. La démonstration du Théorème 0.1 repose sur l'identité (10) et le contrôle du terme $\mathcal{R}_p(u)$ (cf. Lemma 2.2).

1. Introduction

We study the occurrence of blow-up in finite time for radially symmetric solutions to a generalized Smoluchowski–Poisson system

$$\partial_t u = \operatorname{div}\big(a(u)\nabla u - u\nabla v\big) \quad \text{in } (0,\infty)\times B(0,1), \tag{1}$$

$$0 = \Delta v + u - M \quad \text{in } (0,\infty)\times B(0,1), \tag{2}$$

$$\partial_\nu u = \partial_\nu v = 0 \quad \text{on } (0,\infty)\times \partial B(0,1), \tag{3}$$

$$u(0) = u_0 \geqslant 0 \quad \text{in } B(0,1), \qquad \int_{B(0,1)} v(t,x)\,\mathrm{d}x = 0 \quad \text{for any } t \in (0,\infty), \tag{4}$$

where $B(0,1)$ denotes the unit ball of $\mathbb{R}^n$, $n \geqslant 2$, and M the mean value of u_0. The diffusion coefficient a belongs to $\mathcal{C}^2([0,\infty))$ and is assumed to be positive for simplicity (see Remark 2 below). The system (1)–(4) arises in astrophysics as a model of self-gravitating Langevin particles [2] and in biology [6] where it is also known as the parabolic-elliptic Keller–Segel chemotaxis model.

A fundamental property of solutions to (1)–(4) is that

$$u(t) \geqslant 0, \qquad \int_{B(0,1)} u(t,x)\,\mathrm{d}x = M\big|B(0,1)\big|, \quad \text{and} \quad \int_{B(0,1)} v(t,x)\,\mathrm{d}x = 0 \tag{5}$$

for $t > 0$, which readily follows from (1), (3), the comparison principle, the non-negativity of u_0, and the definition of v. It is by now well-known that, if $a \equiv 1$, there are radially symmetric initial data u_0 for which the corresponding solution to (1)–(4) blows up in finite time and this singular phenomenon may happen for any $M > 0$ if $n \geqslant 3$ but only for $M > 8\pi$ if $n = 2$ [5,7]. More recently, it was shown in [4] that there is a critical exponent for the non-linear diffusion coefficient a which separates two different behaviours for the solutions to (1)–(4): on the one hand, if $a(z) \geqslant c(1+z)^\alpha$ and $\alpha > (n-2)/n$, there is a unique global classical solution to (1)–(4) for any non-negative initial condition $u_0 \in L^\infty(B(0,1))$ (and this is actually true for a general smooth bounded domain of $\mathbb{R}^n$, $n \geqslant 1$). On the other hand, if $a(z) \leqslant c(1+z)^\alpha$ and $\alpha < (n-2)/n$, radially symmetric solutions to (1)–(4) blowing up in finite time are constructed in [4].

Except for $n = 2$, the critical case $\alpha = (n-2)/n$ is not covered in [4] and the purpose of this Note is to fill this gap: indeed, the main outcome of our analysis is that, if $a(z) \leqslant c\,(1+z)^{(n-2)/n}$, there are solutions to (1)–(4) blowing up in finite time when M exceeds an explicit threshold. As a by-product of our analysis, we also establish an alternative and simpler proof of the blow-up result in [4] for $\alpha \in [0, (n-2)/n)$. Indeed, the construction of solutions to (1)–(4) blowing up in finite time performed in [4] relies on the possibility of reducing (1)–(4) to a single parabolic equation. The approach used in this paper is completely different and relies on the derivation of a differential inequality of virial type which cannot hold true for all times. When $a \equiv 1$, this technique is used in [7] where it is shown that the moment of order n of u satisfies a differential inequality which contradicts the non-negativity of u after a finite time. Seemingly, the moment of order n of u does not give valuable information when the diffusion is non-linear and we introduce non-linear functions of u to be able to handle this case. We finally point out that the above results are only valid for $n \geqslant 2$: the situation is qualitatively different in the one-dimensional case $n = 1$ and will be considered in a separate paper [3].

2. Finite time blow-up

We first introduce some notation: for $(\alpha, p) \in [0,\infty) \times (1,\infty)$, we define

$$\kappa_p(\alpha) := \frac{(p-1)}{(\alpha+1)(p+\alpha)}\left(\frac{2(n-1)}{p-1}\right)^{\alpha+1}(np)^{(n-2-\alpha n)/n}. \tag{6}$$

We also define $\bar{m}_p(f)$ for $p \geqslant 1$ and $f \in L^\infty(0,1)$ by

$$\bar{m}_p(f) := \frac{1}{p}\int_0^1\left(\int_r^1 f(\rho)\rho^{n-1}\,\mathrm{d}\rho\right)^p r^{n-1}\,\mathrm{d}r. \tag{7}$$

Our main result then reads as follows:

Theorem 2.1. *Assume that there are $\alpha \in [0, (n-2)/n]$, and positive real numbers $c_1 > 0$ and $c_2 > 0$ such that*

$$0 < a(z) \leqslant c_1 z^\alpha + c_2 \quad \text{for } z \geqslant 0. \tag{8}$$

Let $M > 0$ and consider a non-negative radially symmetric function $u_0 \in L^\infty(B(0,1))$ such that $\|u_0\|_1 = M$. Assume further that $E_{M,p}(\bar{m}_p(u_0)) < 0$ for some $p > 1$, where

$$E_{M,p}(z) := c_1\kappa_p(\alpha)\left(\frac{M}{n}\right)^{(2p+n\alpha(p+1))/n} z^{(n-2-\alpha n)/n} + c_2\kappa_p(0)\left(\frac{M}{n}\right)^{(2p)/n} z^{(n-2)/n} + Mz - \frac{1}{p(p+1)}\left(\frac{M}{n}\right)^{p+1}, \quad z \geqslant 0. \tag{9}$$

Then the system (1)–(4) has a unique maximal classical solution (u, v) with finite maximal existence time $T_{\max} \in (0,\infty)$ and $\|u(t)\|_\infty \to \infty$ as $t \to T_{\max}$.

There are initial data u_0 for which $E_{M,p}(\bar{m}_p(u_0)) < 0$ for some $p > 1$. Indeed, observe that $E_{M,p}(0) < 0$ for all $M > 0$ if $\alpha \in [0, (n-2)/n)$ and $n \geqslant 3$, and for $M > \mu_{n,p} := n(c_1 p(p+1)\kappa_p((n-2)/n))^{n/2}$ if $\alpha = (n-2)/n$ and $n \geqslant 2$. It is then sufficient to take u_0 sufficiently concentrated near $x = 0$ so that $\bar{m}_p(u_0)$ is close to zero (for instance, $u_0 = M\delta^{-n}\mathbf{1}_{B(0,\delta)}$ for $\delta > 0$ sufficiently small). Consequently, if $n \geqslant 2$, $\alpha = (n-2)/n$, and $M > \mu_n := \inf_{p>1}\{\mu_{n,p}\}$, there is at least an initial condition u_0 such that $\|u_0\|_1 = M$ and the corresponding solution to (1)–(4) blows up in finite time according to Theorem 2.1. A further question is whether the threshold mass μ_n thus found is sharp. If $n = 2$ ($\alpha = 0$) and $a \equiv 1$, we have $\mu_{2,p} = 4(p+1)$ so that $\mu_2 = 8$ and we recover the well-known threshold condition $\|u_0\|_1 > 8\pi$ for finite time blow-up to occur in the parabolic-elliptic Keller–Segel system [7]. The situation is not so clear in higher space dimensions ($n \geqslant 3$ and $\alpha = (n-2)/n$) as the computation of μ_n is more complicated (observe that $\mu_{n,p} \to \infty$ if $p \to 1$ and $p \to \infty$) and no explicit critical value of the mass guaranteeing global existence seems to be available in the literature. The sharpness of μ_n is then an open question for $n \geqslant 3$. Let us mention at this point that the Cauchy problem for the critical Smoluchowski–Poisson equation is better understood and a critical mass is identified in [1]. However, it is defined as the optimal constant in a functional inequality and thus not explicit, making difficult the comparison with μ_n.

Proof of Theorem 2.1. By [4, Theorem 1.3], there exist a maximal existence time $T_{\max} \in (0, \infty]$ and a unique radially symmetric classical solution $(u, v) \in \mathcal{C}([0, T_{\max}); L^2(B(0,1); \mathbb{R}^2)) \cap \mathcal{C}^{1,2}((0, T_{\max}) \times B(0,1); \mathbb{R}^2)$ to (1)–(4) satisfying (5) for $t \in [0, T_{\max})$. Moreover, if $T_{\max} < \infty$ then $\|u(t)\|_\infty \to \infty$ as $t \to T_{\max}$.

We introduce

$$U(t,r) := \frac{1}{n|B(0,1)|} \int_{B(0,r)} u(t,x)\,\mathrm{d}x \quad \text{and} \quad m_p(t) := \frac{1}{p} \int_0^1 \left(\frac{M}{n} - U(t,r)\right)^p r^{n-1}\,\mathrm{d}r$$

for $(t,r) \in [0, T_{\max}) \times [0,1]$ and derive the following identity of virial type.

Lemma 2.2. *Let A be defined by $A' = a$ and $A(0) = 0$. Then*

$$\frac{\mathrm{d}m_p}{\mathrm{d}t} = M m_p - \frac{1}{p(p+1)} \left(\frac{M}{n}\right)^{p+1} + \mathcal{R}_p(u) \tag{10}$$

with

$$\mathcal{R}_p(u) := \int_0^1 r^{2n-3} \left(\frac{M}{n} - U\right)^{p-2} \left(2(n-1)\left(\frac{M}{n} - U\right) - (p-1) r^n u\right) A(u)\,\mathrm{d}r.$$

Proof. Integrating (1) gives that U solves

$$\partial_t U = r^{n-1} \partial_r A(u) + uU - \frac{M}{n} r^n u \quad \text{with } U(t,0) = \frac{M}{n} - U(t,1) = 0.$$

Consequently,

$$\begin{aligned}
\frac{\mathrm{d}m_p}{\mathrm{d}t} &= \int_0^1 \left(\frac{M}{n} - U\right)^{p-1} \left[\frac{M}{n}(r^n - 1)\partial_r U + \left(\frac{M}{n} - U\right)\partial_r U - r^{2(n-1)} \partial_r A(u)\right] \mathrm{d}r \\
&= -\frac{1}{p}\left(\frac{M}{n}\right)^{p+1} + M m_p + \frac{1}{p+1}\left(\frac{M}{n}\right)^{p+1} \\
&\quad + \int_0^1 \left[2(n-1) r^{2n-3} \left(\frac{M}{n} - U\right)^{p-1} - (p-1) r^{2(n-1)} \left(\frac{M}{n} - U\right)^{p-2} \partial_r U\right] A(u)\,\mathrm{d}r,
\end{aligned}$$

and hence (10). □

The next step is to estimate $\mathcal{R}_p(u)$ in terms of m_p. To this end, we notice that the assumption (8) on a warrants that

$$0 \leqslant A(z) \leqslant \frac{c_1}{1+\alpha} z^{1+\alpha} + c_2 z, \quad z \geqslant 0. \tag{11}$$

In view of (11), we have

$$\begin{aligned}
&\left[2(n-1)\left(\frac{M}{n} - U\right) - (p-1) r^n u\right] A(u) \\
&\quad \leqslant \max\left\{2(n-1)\left(\frac{M}{n} - U\right) - (p-1) r^n u, 0\right\} \left(\frac{c_1}{1+\alpha} u^{1+\alpha} + c_2 u\right) \\
&\quad \leqslant \max\left\{2(n-1)\left(\frac{M}{n} - U\right) - (p-1) r^n u, 0\right\} \left(\frac{c_1}{1+\alpha}\left[\frac{2(n-1)}{(p-1) r^n}\left(\frac{M}{n} - U\right)\right]^\alpha + c_2\right) u
\end{aligned}$$

$$\leqslant \frac{c_1(p-1)}{1+\alpha}\left(\frac{2(n-1)}{p-1}\right)^{1+\alpha}\left(\frac{M}{n}-U\right)^{1+\alpha} r^{-n\alpha}u + 2(n-1)c_2\left(\frac{M}{n}-U\right)u,$$

and thus

$$\mathcal{R}_p(u) \leqslant \frac{c_1(p-1)}{1+\alpha}\left(\frac{2(n-1)}{p-1}\right)^{1+\alpha}\int_0^1 r^{n-2-\alpha n}\left(\frac{M}{n}-U\right)^{p+\alpha-1}\partial_r U\,\mathrm{d}r$$
$$+ 2(n-1)c_2\int_0^1 r^{n-2}\left(\frac{M}{n}-U\right)^{p-1}\partial_r U\,\mathrm{d}r. \tag{12}$$

Since $\alpha \in [0,(n-2)/n]$, the function $r \mapsto r^{(n-2-\alpha n)/n}$ is concave and we infer from the Jensen inequality (with measure $((M/n)-U)^{p+\alpha-1}\partial_r U\,\mathrm{d}r$) that

$$\int_0^1 r^{n-2-\alpha n}\left(\frac{M}{n}-U\right)^{p+\alpha-1}\partial_r U\,\mathrm{d}r$$
$$\leqslant \left(\frac{1}{p+\alpha}\left(\frac{M}{n}\right)^{p+\alpha}\right)^{(2+\alpha n)/n}\left(\int_0^1 r^n\left(\frac{M}{n}-U\right)^{p+\alpha-1}\partial_r U\,\mathrm{d}r\right)^{(n-2-\alpha n)/n}$$
$$\leqslant \left(\frac{1}{p+\alpha}\left(\frac{M}{n}\right)^{p+\alpha}\right)^{(2+\alpha n)/n}\left(\frac{n}{p+\alpha}\int_0^1 r^{n-1}\left(\frac{M}{n}-U\right)^{p+\alpha}\mathrm{d}r\right)^{(n-2-\alpha n)/n}$$
$$\leqslant \frac{(np)^{(n-2-\alpha n)/n}}{p+\alpha}\left(\frac{M}{n}\right)^{(2p+\alpha n(p+1))/n} m_p^{(n-2-\alpha n)/n}.$$

Arguing in a similar way to estimate the second integral in the right-hand side of (12), we deduce from (12) that

$$\mathcal{R}_p(u) \leqslant E_{M,p}(m_p) - Mm_p + \frac{1}{p(p+1)}\left(\frac{M}{n}\right)^{p+1}.$$

Inserting this estimate in (10) we arrive at

$$\frac{\mathrm{d}m_p(t)}{\mathrm{d}t} \leqslant E_{M,p}\big(m_p(t)\big) \quad \text{for } t \in [0,T_{\max}). \tag{13}$$

Assume now for contradiction that $T_{\max} = \infty$. Since $z \mapsto E_{M,p}(z)$ is an increasing function and $m_p(0) = \bar{m}_p(u_0)$, we realize that, as soon as $E_{M,p}(m_p(0)) < 0$, we have $m_p(t_0) = 0$ for some $t_0 \in (0,\infty)$. Thus, $U(t_0,r) = M/n$ for all $r \in [0,1]$ which contradicts the fact that $U(t_0,0) = 0$. Consequently, $T_{\max} < \infty$ and the proof is complete. □

Remark 1. If $a \equiv 1$ (i.e. $\alpha = 0$), Lemma 2.2 is also valid for $p = 1$ and m_1 coincides with the moment of order n used in [7].

Remark 2. The requirement $a > 0$ is only used to have classical solutions to (1)–(4) but does not play any role in the blow-up condition and the identity of virial type (Lemma 2.2). Thus, Theorem 2.1 remains valid if the diffusion is degenerate (for instance, $a(r) = mr^{m-1}$ with $m \in [1, 2(n-1)/n]$) provided an appropriate notion of weak solutions is available.

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