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Group Theory

Baer–Suzuki theorem for the solvable radical of a finite group

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Abstract

We prove that an element g of prime order $q > 3$ belongs to the solvable radical $\mathfrak{R}(G)$ of a finite group if and only if for every $x \in G$ the subgroup generated by g and xgx^{-1} is solvable. This theorem implies that a finite group G is solvable if and only if in each conjugacy class of G every two elements generate a solvable subgroup. ***To cite this article: N. Gordeev et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Le théorème de Baer–Suzuki pour le radical résoluble d'un groupe fini. Nous démontrons qu'un élément g d'ordre premier $q > 3$ appartient au radical résoluble $\mathfrak{R}(G)$ d'un groupe fini G si et seulement si pour tout $x \in G$ le sous-groupe engendré par x et xgx^{-1} est résoluble. Ce théorème implique qu'un groupe fini G est résoluble si et seulement si dans chaque classe de conjugaison de G tout couple d'éléments engendre un sous-groupe résoluble. ***Pour citer cet article : N. Gordeev et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Le théorème classique de Baer–Suzuki [2,27,1] s'énonce :

Théorème 0.1 *(Baer–Suzuki). Le radical nilpotent d'un groupe fini G est l'ensemble des éléments $g \in G$ tels que pour tout $a \in G$ le sous-groupe engendré par g et aga^{-1} est nilpotent.*

Quelques travaux récents [6,7,9,10], ont essayé de décrire de façon analogue le radical résoluble d'un groupe fini. En particulier, le problème suivant est parallèle au résultat de Baer–Suzuki :

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Problème 0.1. Soit G un groupe fini de radical résoluble $\mathfrak{R}(G)$. Quel est le nombre minimal k tel que $g \in \mathfrak{R}(G)$ si et seulement si le sous-groupe engendré par $x_1 g x_1^{-1}, \dots, x_k g x_k^{-1}$ est résoluble pour toute partie à k éléments $\{x_1, \dots, x_k\} \subset G$?

Récemment on a démontré [11] :

Théorème 0.2. *Le radical résoluble d'un groupe fini G est l'ensemble des éléments g satisfaisant à la propriété suivante : pour toute partie à* 3 *éléments* $\{a, b, c\} \subset G$*, le sous-groupe engendré par les éléments* $g, aga^{-1}, bgb^{-1}, cgc^{-1}$ *est résoluble.*

Donc un élément d'un groupe fini appartient au radical résoluble si et seulement si toute partie à 4 éléments de sa classe de conjugaison engendre un groupe résoluble. Cet énoncé peut être vu comme un théorème de type Baer–Suzuki par rapport à la propriété de résolubilité. Le Théorème 0.2 est le meilleur possible : dans les groupes symétriques S_n $(n \geqslant 5)$, tout triplet de transpositions engendre un sous-groupe résoluble.

Cependant, comme indiqué par Flavell [6], on peut espérer qu'un analogue précis du théorème de Baer–Suzuki soit valable pour les éléments d'ordre premier > 3 dans $\mathfrak{R}(G)$. Notre résultat principal confirme cette espoir :

Théorème 0.3. *Soit G un groupe fini. Alors un élément g d'ordre premier* $q > 3$ *appartient à* $\mathfrak{R}(G)$ *si et seulement si pour tout* $x \in G$ *le sous-groupe* $H = \langle g, xgx^{-1} \rangle$ *est résoluble.*

Prenant en compte le « théorème $p^\alpha q^\beta$ » de Burnside, le Théorème 0.3 implique :

Corollaire 0.4. *Un groupe fini G est résoluble si et seulement si dans chaque classe de conjugaison de G tout couple d'éléments engendre un sous-groupe résoluble.*

Remarque 0.5. Le Théorème 0.3 et le Corollaire 0.4 restent vrais pour les groupes linéaires (non nécessairement finis).

Remarque 0.6. La démonstration du Théorème 0.3 utilise la classification des groupes finis simples. On peut démontrer le Corollaire 0.4 sans utiliser la classification grâce à la caractérisation de J. Thompson des groupes non-résolubles minimaux [28,5]. Flavell a réussi à démontrer, sans utiliser la classification, un analogue du Théorème 0.2 pour $k = 10$ [6] et aussi le Théorème 0.2 sous l'hypothèse supplémentaire que l'ordre de $g \in G$ est un nombre premier > 3.

Remarque 0.7. R. Guralnick nous a informé que les Théorèmes 0.3 et 0.2 ont été démontrés indépendamment dans des travaux à paraître par Guest, Guralnick et Flavell [16,17,8]. De plus, Flavell [8] a réduit le nombre k à 7, la démonstration n'utilisant pas la classification.

1. Main result

The classical Baer–Suzuki theorem [2,27,1] states that:

Theorem 1.1 *(Baer–Suzuki). The nilpotent radical of a finite group G coincides with the collection of* $g \in G$ *satisfying the property*: *for any* $a \in G$ *the subgroup generated by g and* aga^{-1} *is nilpotent.*

Within past few years a lot of efforts have been made in order to describe the solvable radical of a finite group and to establish a sharp analogue of the Baer–Suzuki theorem with respect to the solvability property (see [6,7,9,10]). In particular, the following problem is parallel to the Baer–Suzuki result:

Problem 1.2. Let G be a finite group with the solvable radical $\mathfrak{R}(G)$. What is the minimal number k such that $g \in \mathfrak{R}(G)$ if and only if the subgroup generated by $x_1 g x_1^{-1}, \dots, x_k g x_k^{-1}$ is solvable for any $x_1, \dots, x_k$ in G?

Recently (see [11]) it was proved that

Theorem 1.3. *The solvable radical of a finite group G coincides with the collection of $g \in G$ satisfying the property: for any 3 elements $a, b, c \in G$ the subgroup generated by the conjugates $g, aga^{-1}, bgb^{-1}, cgc^{-1}$ is solvable.*

Thus, an element of a finite group belongs to the solvable radical if and only if *any four* its conjugates generate a solvable group. This statement may be viewed as a theorem of Baer–Suzuki type with respect to the solvability property. Theorem 1.3 is sharp and provides the best possible characterization: in the symmetric groups S_n ($n \geqslant 5$) any triple of transpositions generates a solvable subgroup.

However, as mentioned by Flavell [6], one can expect a precise analogue of the Baer–Suzuki theorem to hold for the elements of prime order greater than 3 in $\mathfrak{R}(G)$. Our main result confirms this expectation:

Theorem 1.4. *Let G be a finite group. An element g of prime order $q > 3$ belongs to $\mathfrak{R}(G)$ if and only if for any $x \in G$ the subgroup $H = \langle g, xgx^{-1} \rangle$ is solvable.*

Theorem 1.4 together with Burnside's $p^\alpha q^\beta$-theorem implies

Corollary 1.5. *A finite group G is solvable if and only if in each conjugacy class of G every two elements generate a solvable subgroup.*

Remark 1.6. Theorem 1.4 and Corollary 1.5 remain true for the linear groups (not necessarily finite).

Remark 1.7. The proof of Theorem 1.4 uses the classification of finite simple groups (CFSG). The proof of Corollary 1.5 can be obtained without classification using J. Thompson's characterization of the minimal nonsolvable groups [28,5]. Flavell managed to prove, without CSFG, an analogue of Theorem 1.3 for $k = 10$ [6] and Theorem 1.3 under the additional assumption that $g \in G$ is of prime order $q > 3$.

Remark 1.8. R. Guralnick informed us that Theorems 1.4 and 1.3 were independently proved in forthcoming works by Guest, Guralnick, and Flavell [16,17,8]. Flavell [8] reduced k to 7 with a proof which does not rely on CFSG.

Notational conventions. Whenever possible, we maintain the notation of [10] which mainly follows [25,3]. In particular, we adopt the notation of [3] for twisted forms of Chevalley groups (so unitary groups are denoted by $PSU_n(q^2)$ and not by $PSU_n(q)$). However, the classification of outer automorphisms follows [14, p. 60], [13, p. 78]. We say that a finite group G is almost simple if it contains a unique normal simple group L such that $L \leqslant G \leqslant \mathrm{Aut}(L)$.

2. Outline of the proof

We reduce Theorem 1.4 to the following statement:

Theorem 2.1. *Let G be a finite almost simple group, and let $g \in G$ be of prime order > 3. Then there is $x \in G$ such that the subgroup generated by g and xgx^{-1} is not solvable.*

The reduction is fairly standard and follows the scheme of [9,10], so the rest of the paper is devoted to the outline of the proof of Theorem 2.1. We refer to the property stated in Theorem 2.1 as Property $(*)$. Throughout this section, G stands for a finite almost simple group, $L \leqslant G \leqslant \mathrm{Aut}\,L$. We use CFSG to prove that G satisfies $(*)$.

2.1. Alternating groups, sporadic groups, groups of Lie rank 1 and $^2F_4$

Lemma 2.2. *Let $L = A_n$, $n \geqslant 5$, be an alternating group. Then G satisfies $(*)$.*

Proof. Clearly it is enough to consider the alternating groups: as $\mathrm{Aut}(A_n) = S_n$ for $n \neq 6$ and $[\mathrm{Aut}(A_6) : A_6] = 4$, any element of odd order in $\mathrm{Aut}(A_n)$ lies in A_n. For $G = A_n$, $n \geqslant 5$, the statement is easily proved by induction. □

Lemma 2.3. *Let L be a sporadic simple group. Then G satisfies $(*)$.*

Proof. As the group of outer automorphisms of any sporadic group is of order at most 2, it is enough to treat the case where G is a simple sporadic group. Here the proof goes, word for word, as in [9, Prop. 9.1]. Namely, case-by-case analysis shows that any element $x \in G$ of prime order $q > 3$ is either contained in a smaller simple subgroup of G or its normalizer is a maximal subgroup of G. In the latter case it is enough to conjugate g by an element x not belonging to $N_G(\langle g\rangle)$ to ensure that $\langle g, xgx^{-1}\rangle = G$. $\square$

Proposition 2.4. *Suppose that the Lie rank of L is 1. Then G satisfies* $(*)$.

Proof. Let $g \in G$ be of prime order > 3. We shall check that there is $x = g^\alpha \in G$, $\alpha \in L$, such that the subgroup of G generated by g and x is not solvable.

If $L = PSL_2(q)$, $q \geqslant 4$, the result follows from [18, Lemma 3.1]. If $L = PSU_3(q^2)$, $q > 2$, the result follows from the proof of [18, Lemma 3.3] with the single exception where g is a field automorphism. In the latter case we can take g to be a standard one. It normalizes a subgroup of type A_1 generated by a self-conjugate root of A_2, and the result follows from [18, Lemma 3.1].

If L is a Suzuki group ${}^2B_2(2^{2m+1})$, $m \geqslant 1$, or a Ree group ${}^2G_2(3^{2m+1})$, $m > 0$, we only have to consider the cases where $g \in G$ is a semisimple element of L (since g is of prime order > 3 and cannot be unipotent), or a field automorphism (since every outer automorphism is a field automorphism).

1. *Suzuki case, g is semisimple*. Suppose $g \in {}^2B_2(2^{2m+1})$ is a semisimple element of order greater than 3. Then g is regular [3,19]. Choose a cyclic torus T in ${}^2B_2(2^{2m+1})$ of order $q_0^2 + \sqrt{2}q_0 + 1$, where $q_0 = 2^m\sqrt{2}$. It follows from the description of maximal subgroups of L (see [26]) that $N_L(T)$ is maximal in L. Moreover, $N_L(T)/T = \mathbb{Z}_4$ (cf. [24,29,22]).

Suppose $g \notin T$. Since the order of g is bigger than 3 and g is regular, any element $x = g^\alpha$, $\alpha \in L$ conjugate to g does not belong to $N_L(T)$. There exists $z \in L$ such that $[x, z]$ is a generator of a nonsplit torus T [15]. Then $\langle x, zxz^{-1}\rangle$ contains the nonsplit torus T and the element x which does not belong to $N_L(T)$. Correspondingly, $\langle g, z_1gz_1^{-1}\rangle = L$, for some $z_1 \in L$ because of maximality of $N_L(T)$.

Let now $g \in T$. Fix a maximal cyclic torus T_1 in ${}^2B_2(2^{2m+1})$ of order $q_0^2 - \sqrt{2}q_0 + 1$, where $q_0 = 2^m\sqrt{2}$. Again $N_L(T_1)$ is maximal in L, and $N_L(T_1)/T_1 = \mathbb{Z}_4$ (cf. [24,22]). Then g does not belong to T_1, and we repeat the above arguments.

2. *Ree case, x is semisimple.* Suppose $g \in {}^2G_2(2^{2m+1})$ is a semisimple element of order greater than 3. Then g is regular [3,19]. Let us choose cyclic tori T, T_1 in ${}^2G_2(3^{2m+1})$ of orders $q_0^2 \pm \sqrt{3}q_0 + 1$, where $q_0 = 3^m\sqrt{3}$. Then $N_L(T)$, $N_L(T_1)$ are maximal in L, and $N_L(T)/T = N_L(T_1)/T_1 = \mathbb{Z}_6$ (cf. [21,20,29,22]). With these data the proof literally repeats the proof for the Suzuki case.

3. Let g be a nontrivial field automorphism of L of prime order greater than 3. For the cases $L = PSL_2(q)$, $q \geqslant 4$, or $L = {}^2B_2(2^{2m+1})$, $m \geqslant 1$, we use a slightly modified counting method from [18, Lemma 3.1].

The Ree groups ${}^2G_2(3^{2m+1})$, $m > 0$, are not minimal and contain a subgroup isomorphic to $PSL_2(q)$, normalized but not centralized by g. Thus there exists $y = g^\alpha \in L$, $\alpha \in L$, such that $\langle g, y\rangle$ is not solvable. $\square$

Lemma 2.5. *Let L be a group of type ${}^2F_4$. Then G satisfies* $(*)$.

Proof. Let $g \in G$ be a unipotent element of prime order > 3. Then it is easy to reduce the problem to the case of Lie rank 1. The same idea works if g is a field automorphism of L.

So we can assume that g is a semisimple element which is not contained in a proper parabolic subgroup and thus lies in one of the tori (4)–(8) from the list of [23]. Suppose that g does not belong to T_{11}. Since $|N_G(T_{11})/T_{11}| = 12$, g does not belong to $N_G(T_{11})$. There exists t_1 such that $[t_1, g^\alpha]$, $\alpha \in L$, is a generator of the cyclic nonsplit torus $H = T_{11}$ (see [15]). Then $\langle g^\alpha, t_1g^\alpha t_1^{-1}\rangle$ contains the nonsplit torus H and the element g^α which does not belong to $N_G(H)$. Thus $\langle g, s_1gs_1^{-1}\rangle = L$ for some $s_1 \in L$. If g belongs to T_{11} we choose t_2 such that $[t_2, g^\beta]$, $\beta \in L$, is a generator of the torus $H = T_{10}$. Now we repeat the previous arguments. $\square$

2.2. *General case*

Let G be a finite almost simple group, $L \leqslant G \leqslant \operatorname{Aut} L$, the Lie rank of L is $\geqslant 2$, $L \neq {}^2F_4$. This is the main part of the proof of Theorem 1.4. In contrast with the rank 1 case we avoid here considerations related to the specific subgroup structure of the groups in question and use some basic results on algebraic groups instead.

Let $g \in G$ be of prime order $q > 3$. Our aim is to prove property $*$ using a kind of induction by parabolic subgroups of G and the corresponding Levi subgroups. In order to use such an induction, we have to embed G into a reductive algebraic group. For the sake of convenience, we replace induction by studying the minimal counterexample to $*$.

If $g \in G$ is a field automorphism of L, then it normalizes but does not centralize a smaller rank group, and we are done. Hence it is enough to prove the following:

Theorem 2.6. *Let $G = G(K)$ be a simple group of Lie type,* $\mathrm{rank}(G) \geqslant 2$*, K a finite field,* $\mathrm{char}(K) = p$. *Let σ be a diagonal automorphism of G, and let $\Gamma = \langle \sigma, G\rangle$. Let $g \in \Gamma$ be of prime order $q > 3$. Then there exists $x \in \Gamma$ such that the group $\langle g, xgx^{-1}\rangle$ is not solvable.*

First we prove that for every pair $\langle \sigma, G\rangle$ there exists a reductive group $\mathfrak{G}$ over K satisfying the following conditions:

- the derived group $\mathfrak{G}'$ is simply connected;
- $\mathfrak{G}'(K)/Z(\mathfrak{G}'(K)) = G$;
- there is $\tau \in \mathfrak{G}(K)$ such that $\langle \tau, \mathfrak{G}'(K)\rangle / Z(\langle \tau, \mathfrak{G}'(K)\rangle) = \langle \sigma, G\rangle$.

Let G_{sc} be a simple, simply connected algebraic group over a finite field K, and let G_{ad} be the corresponding adjoint group. Then $G = [G_{ad}(K), G_{ad}(K)] = G_{sc}(K)/Z(G_{sc}(K))$.

Let B be a Borel subgroup of $G_{ad}(K)$, and let $g \in B$. There exists $y \in G$ such that $ygy^{-1} = u_\alpha u$, where $u_\alpha \neq 1$ is a unipotent element corresponding to a simple root α. Using the rank one case, we get x such that $\langle g, xgx^{-1}\rangle$ is not solvable.

Thus we can assume that g is a semisimple element which does not lie in any Borel subgroup.

Let $\Gamma = \langle \sigma, G\rangle$ be a minimal counter-example to Theorem 2.6, and let $g \in \Gamma$.

Using arguments from [12] and [4], one can prove that g is of the form $g = v\dot{w}_c$ where w_c is the Coxeter element of the Weyl group of $G_{ad}(K)$ and $v \in U \leqslant G$. In particular, this implies that g is regular.

By [10], for any $g = v\dot{w}_c$ there is $x \in G$ such that $[g, x] = u \in U$.

Put $H = \langle g, xgx^{-1}\rangle$. Suppose H is solvable. Denote by H_{pq} a pq-Hall subgroup of H, and let A be a maximal abelian normal subgroup of H_{pq}. Denote by A_p the p-Sylow subgroup of A. One can prove that g does not normalize any unipotent subgroup of G. This, in particular, implies that $A_p = 1$, because A_p is normalized by g. Hence $|A| = q^s$.

The next fact about the structure of the minimal counterexample is as follows. Let $F \lneq G_{ad}$ be a reductive subgroup defined over K, and suppose $g \in F = F(K)$. Consider F^0, the identity component of F. Then F^0 is a K-defined connected reductive group. Suppose that F^0 is not a torus. Then $g \in F$ leads to a contradiction and therefore *g cannot lie in any proper reductive subgroup of G_{ad}* other than a torus. This leads to a crucial reduction:

Proposition 2.7. *If Γ is a minimal counter-example, $g \in \Gamma$, then the reductive group $\mathfrak{G}$ corresponding to G is either the linear group GL_q or the unitary group U_q.*

Each of these cases is considered separately and in both of them it turns out that g is represented by a monomial matrix, which leads to a contradiction by Lemma 2.2.

Remark 2.8. It is possible to give another proof of the main theorem which uses the structure of large cyclic anisotropic tori, description of maximal subgroups in groups of Lie type and Gow's theorem [15]. An example of such a proof is given in Section 2.1 for groups of type ${}^2F_4$ and groups of Lie rank 1.

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