Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 347 (2009) 119–122

COMPTES RENDUS MATHEMATIQUE

Number Theory

Means of algebraic numbers in the unit disk

Igor E. Pritsker [1]

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA

Received 19 November 2008; accepted after revision 9 January 2009

Available online 6 February 2009

Presented by Jean-Pierre Serre

Abstract

Schur studied limits of the arithmetic means s_n of zeros for polynomials of degree n with integer coefficients and simple zeros in the closed unit disk. If the leading coefficients are bounded, Schur proved that $\limsup_{n\to\infty} |s_n| \leqslant 1 - \sqrt{\mathrm{e}}/2$. We show that $s_n \to 0$, and estimate the rate of convergence by generalizing the Erdős–Turán theorem on the distribution of zeros. ***To cite this article: I.E. Pritsker, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Moyennes de nombres algébriques dans le disque unité. Schur a étudié les limites des moyennes arithmétiques s_n des zéros pour les polynômes à coefficients entiers de degré n ayant des zéros simples dans le disque unité fermé. Lorsque les coefficients dominants restent bornés, Schur a démontré que $\limsup_{n\to\infty} |s_n| \leqslant 1 - \sqrt{\mathrm{e}}/2$. Nous prouvons que $s_n \to 0$. Nous donnons une estimation du taux de convergence, grâce à une généralisation d'un théorème de Erdős–Turán sur la distribution des zéros. ***Pour citer cet article : I.E. Pritsker, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Schur's problem and equidistribution of zeros

Let $\mathbb{Z}_n(D)$ be the set of polynomials of degree n with integer coefficients and all zeros in the closed unit disk D. We denote the subset of $\mathbb{Z}_n(D)$ with simple zeros by $\mathbb{Z}_n^1(D)$. Given $M > 0$, we write $P_n = a_n z^n + \cdots \in \mathbb{Z}_n^1(D, M)$ if $|a_n| \leqslant M$ and $P_n \in \mathbb{Z}_n^1(D)$ (respectively $P_n \in \mathbb{Z}_n(D, M)$ if $|a_n| \leqslant M$ and $P_n \in \mathbb{Z}_n(D)$). Schur [9, §8] studied the limiting behavior of the arithmetic means s_n of zeros for polynomials from $\mathbb{Z}_n^1(D, M)$ as $n \to \infty$, where $M > 0$ is an arbitrary fixed number. He showed that $\limsup_{n\to\infty} |s_n| \leqslant 1 - \sqrt{\mathrm{e}}/2$, and remarked that this $\limsup$ is equal to 0 for *monic* polynomials from $\mathbb{Z}_n(D)$ by Kronecker's theorem [6]. We prove that $\lim_{n\to\infty} s_n = 0$ for any sequence of polynomials from Schur's class $\mathbb{Z}_n^1(D, M)$, $n \in \mathbb{N}$. This result is obtained as a consequence of the asymptotic equidistribution of zeros near the unit circle. Namely, if $\{\alpha_k\}_{k=1}^n$ are the zeros of P_n, we define the counting measure $\tau_n := \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}$, where δ_{α_k} is the unit point mass at α_k. Consider the normalized arclength measure μ on the unit

E-mail address: igor@math.okstate.edu.

URL: http://www.math.okstate.edu/~igor/.

[1] Research was partially supported by NSA, and by the Alexander von Humboldt Foundation.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.01.002

circumference $\mathbb{T}$, with $\mathrm{d}\mu(\mathrm{e}^{\mathrm{i}t}) := \frac{1}{2\pi}\,\mathrm{d}t$. If the τ_n converge weakly to μ as $n \to \infty$ ($\tau_n \xrightarrow{*} \mu$) then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \int z\,\mathrm{d}\tau_n(z) = \int z\,\mathrm{d}\mu(z) = 0$. Thus Schur's problem is solved by the following result.

Theorem 1.1. *If $P_n(z) = a_n z^n + \cdots \in \mathbb{Z}_n^1(D),\ n \in \mathbb{N}$, satisfy $\lim_{n\to\infty} |a_n|^{1/n} = 1$, then $\tau_n \xrightarrow{*} \mu$ as $n \to \infty$.*

Ideas on the equidistribution of zeros date back to Jentzsch and Szegő, cf. [1, Ch. 2]. They were developed further by Erdős and Turán [4], and many others; see [1] for history and additional references. More recently, this topic received renewed attention in number theory, e.g. in the work of Bilu [2]. If the leading coefficients of polynomials are bounded, then we can allow even certain multiple zeros. Define the multiplicity of an irreducible factor Q of P_n as the integer $m_n \geqslant 0$ such that Q^{m_n} divides P_n, but Q^{m_n+1} does not divide P_n. If a factor Q occurs infinitely often in a sequence $P_n,\ n \in \mathbb{N}$, then $m_n = \mathrm{o}(n)$ means $\lim_{n\to\infty} m_n/n = 0$. If Q is present only in finitely many P_n, then $m_n = \mathrm{o}(n)$ by definition.

Theorem 1.2. *Assume that $P_n \in \mathbb{Z}_n(D, M),\ n \in \mathbb{N}$. If every irreducible factor in the sequence of polynomials P_n has multiplicity $\mathrm{o}(n)$, then $\tau_n \xrightarrow{*} \mu$ as $n \to \infty$.*

Corollary 1.3. *If $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k),\ n \in \mathbb{N}$, satisfy the assumptions of Theorem 1.1 or 1.2, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k^m = 0, \quad m \in \mathbb{N}.$$

We also show that the norms $\|P_n\|_\infty := \max_{|z|=1} |P_n(z)|$ have at most subexponential growth.

Corollary 1.4. *If $P_n,\ n \in \mathbb{N}$, satisfy the assumptions of Theorem 1.1 or Theorem 1.2, then*

$$\lim_{n\to\infty} \|P_n\|_\infty^{1/n} = 1.$$

This result is somewhat unexpected, as we have no direct control of the norm or coefficients (except for the leading one). For example, $P_n(z) = (z-1)^n$ has norm $\|P_n\|_\infty = 2^n$.

We now consider quantitative aspects of the convergence $\tau_n \xrightarrow{*} \mu$. As an application, we obtain estimates of the convergence rate of s_n to 0 in Schur's problem. A classical result on the distribution of zeros is due to Erdős and Turán [4]. For $P_n(z) = \sum_{k=0}^n a_k z^k$ with $a_k \in \mathbb{C}$, let $N(\phi_1, \phi_2)$ be the number of zeros in the sector $\{z \in \mathbb{C}\colon\ 0 \leqslant \phi_1 \leqslant \arg(z) \leqslant \phi_2 < 2\pi\}$, where $\phi_1 < \phi_2$. Erdős and Turán [4] proved that

$$\left| \frac{N(\phi_1, \phi_2)}{n} - \frac{\phi_2 - \phi_1}{2\pi} \right| \leqslant 16 \sqrt{\frac{1}{n} \log \frac{\|P_n\|_\infty}{\sqrt{|a_0 a_n|}}}. \tag{1}$$

The constant 16 was improved by Ganelius, and $\|P_n\|_\infty$ was replaced by weaker integral norms by Amoroso and Mignotte; see [1] for more history and references. Our main difficulty in applying (1) to Schur's problem is the absence of an effective estimate for $\|P_n\|_\infty,\ P_n \in \mathbb{Z}_n^1(D, M)$. We prove a new "discrepancy" estimate via energy considerations from potential theory. These ideas originated in part in the work of Kleiner, and were developed by Sjögren and Hüsing, see [1, Ch. 5]. We also use the Mahler measure of a polynomial $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k)$, defined by $M(P_n) := \exp(\frac{1}{2\pi} \int_0^{2\pi} \log|P_n(\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t)$. Note that $M(P_n) = \lim_{p\to 0} \|P_n\|_p$, where $\|P_n\|_p := (\frac{1}{2\pi} \int_0^{2\pi} |P_n(\mathrm{e}^{\mathrm{i}t})|^p\,\mathrm{d}t)^{1/p},\ p > 0$. Jensen's formula readily gives $M(P_n) = |a_n| \prod_{k=1}^n \max(1, |\alpha_k|)$ [3, p. 3]. Hence $M(P_n) = |a_n| \leqslant M$ for any $P_n \in \mathbb{Z}_n(D, M)$.

Theorem 1.5. *Let $\phi : \mathbb{C} \to \mathbb{R}$ satisfy $|\phi(z) - \phi(t)| \leqslant A|z - t|,\ z, t \in \mathbb{C}$, and $\operatorname{supp}(\phi) \subset \{z\colon\ |z| \leqslant R\}$. If $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k)$ is a polynomial with integer coefficients and simple zeros, then*

$$\left| \frac{1}{n} \sum_{k=1}^{n} \phi(\alpha_k) - \int \phi\,\mathrm{d}\mu \right| \leqslant A(2R+1) \sqrt{\frac{\log \max(n, M(P_n))}{n}}, \quad n \geqslant 55. \tag{2}$$

This theorem is related to recent results of Favre and Rivera-Letelier [5], obtained in a different setting. Choosing ϕ appropriately, we obtain an estimate of the means s_n in Schur's problem.

Corollary 1.6. *If $P_n \in \mathbb{Z}_n^1(D, M)$ then*

$$\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right| \leqslant 8\sqrt{\frac{\log n}{n}}, \quad n \geqslant \max(M, 55).$$

We also have an improvement of Corollary 1.4 for Schur's class $\mathbb{Z}_n^1(D, M)$.

Corollary 1.7. *If $\{P_n\}_{n=1}^{\infty} \in \mathbb{Z}_n^1(D, M)$ then there is some $c > 0$ such that $\|P_n\|_\infty \leqslant \mathrm{e}^{c\sqrt{n}\log n}$ as $n \to \infty$.*

The proof of Theorem 1.5 gives a result for arbitrary polynomials with simple zeros, and for any continuous ϕ with finite Dirichlet integral $D[\phi] = \iint(\phi_x^2 + \phi_y^2)\,\mathrm{d}A$. Moreover, all arguments may be extended to general sets of logarithmic capacity 1, e.g. to $[-2, 2]$. Using the characteristic function $\phi = \chi_E$, we can prove general discrepancy estimates on arbitrary sets, and obtain an Erdős–Turán-type theorem. Our results have a number of applications to the problems on integer polynomials considered in [3].

2. Proofs

Proof of Theorem 1.1. Observe that the discriminant $\Delta(P_n) := a_n^{2n-2}\prod_{1\leqslant j<k\leqslant n}(\alpha_j - \alpha_k)^2$ is an integer, as a symmetric form in the zeros of P_n. Since P_n has simple roots, we have $\Delta(P_n) \neq 0$ and $|\Delta(P_n)| \geqslant 1$. Using weak compactness, we assume that $\tau_n \xrightarrow{*} \tau$, where τ is a probability measure on D. Let $K_M(x,t) := \min(-\log|x-t|, M)$. Since $\tau_n \times \tau_n \xrightarrow{*} \tau \times \tau$, we obtain for the energy of τ that

$$\begin{aligned} I[\tau] &:= -\iint \log|x-t|\,\mathrm{d}\tau(x)\,\mathrm{d}\tau(t) = \lim_{M\to\infty}\left(\lim_{n\to\infty}\iint K_M(x,t)\,\mathrm{d}\tau_n(x)\,\mathrm{d}\tau_n(t)\right) \\ &= \lim_{M\to\infty}\left(\lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{j\neq k}K_M(\alpha_j,\alpha_k) + \frac{M}{n}\right)\right) \leqslant \lim_{M\to\infty}\left(\liminf_{n\to\infty}\frac{1}{n^2}\sum_{j\neq k}\log\frac{1}{|\alpha_j-\alpha_k|}\right) \\ &= \liminf_{n\to\infty}\frac{1}{n^2}\log\frac{|a_n|^{2n-2}}{\Delta(P_n)} \leqslant \liminf_{n\to\infty}\frac{1}{n^2}\log|a_n|^{2n-2} = 0. \end{aligned}$$

Thus $I[\tau] \leqslant 0$. But $I[\nu] > 0$ for any probability measure ν on D, except for μ [7]. Hence $\tau = \mu$. □

Proof of Theorem 1.2. Let $\phi \in C(\mathbb{C})$. Note that for any $\epsilon > 0$ there are finitely many irreducible factors Q in the sequence P_n such that $|\int\phi\,\mathrm{d}\tau(Q) - \int\phi\,\mathrm{d}\mu| \geqslant \epsilon$, where $\tau(Q)$ is the zero counting measure for Q. Indeed, if we have an infinite sequence of such Q_m, then $\deg(Q_m) \to \infty$, as there are only finitely many $Q_m \in \mathbb{Z}_n(D, M)$ of bounded degree. Hence $\int\phi\,\mathrm{d}\tau(Q_m) \to \int\phi\,\mathrm{d}\mu$ by Theorem 1.1. Let the number of such exceptional factors Q_m be N. Then we have $|n\int\phi\,\mathrm{d}\tau_n - n\int\phi\,\mathrm{d}\mu| \leqslant N\mathrm{o}(n)\max_D|\phi - \int\phi\,\mathrm{d}\mu| + (n-N)\epsilon,\ n \in \mathbb{N}$. Hence $\limsup_{n\to\infty}|\int\phi\,\mathrm{d}\tau_n - \int\phi\,\mathrm{d}\mu| \leqslant \epsilon$, and $\lim_{n\to\infty}\int\phi\,\mathrm{d}\tau_n = \int\phi\,\mathrm{d}\mu$ after letting $\epsilon \to 0$. □

Proof of Corollary 1.3. Let $\phi(z) = z^m$ and write $\lim_{n\to\infty}\int z^m\,\mathrm{d}\tau_n(z) = \int z^m\,\mathrm{d}\mu(z) = 0$. □

Proof of Corollary 1.4. Let $\|P_n\|_\infty = |P_n(z_n)|,\ z_n \in D$, and assume $\lim_{n\to\infty} z_n = z_0 \in D$ by compactness. Then $\|P_n\|_\infty = \exp(\log|P_n(z_n)|) = |a_n|\exp(n\int\log|z_n - t|\,\mathrm{d}\tau_n(t))$. Since $\tau_n \xrightarrow{*} \mu$, Theorem I.6.8 of [8] gives $\limsup_{n\to\infty}\|P_n\|_\infty^{1/n} \leqslant \exp(\int\log|z_0 - t|\,\mathrm{d}\mu(t)) = 1$ [8, p. 22]. But $\|P_n\|_\infty \geqslant |a_n| \geqslant 1$, see [1, p. 16]. □

Proof of Theorem 1.5. Given $r > 0$, define the measures ν_k^r with $\mathrm{d}\nu_k^r(\alpha_k + r\mathrm{e}^{\mathrm{i}t}) = \mathrm{d}t/(2\pi),\ t \in [0, 2\pi)$. Let $\tau_n^r := \frac{1}{n}\sum_{k=1}^n \nu_k^r$, and estimate $|\int\phi\,\mathrm{d}\tau_n - \int\phi\,\mathrm{d}\tau_n^r| \leqslant \frac{1}{n}\sum_{k=1}^n\frac{1}{2\pi}\int_0^{2\pi}|\phi(\alpha_k) - \phi(\alpha_k + r\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t \leqslant \omega_\phi(r)$, where $\omega_\phi(r) := \sup_{|z-\zeta|\leqslant r}|\phi(z) - \phi(\zeta)|$ is the modulus of continuity of ϕ.

Let $p_\nu(z) := -\int \log|z-t|\,\mathrm{d}\nu(t)$ be the potential of a measure ν. A direct evaluation gives that $p_{\nu_k^r}(z) = -\log\max(r, |z-\alpha_k|)$ and $p_\mu(z) = -\log\max(1, |z|)$ [8, p. 22]. Consider $\sigma := \tau_n^r - \mu$, $\sigma(\mathbb{C}) = 0$. One computes (or see [8, p. 92]) that $\mathrm{d}\sigma = -\frac{1}{2\pi}(\partial p_\sigma/\partial n_+ + \partial p_\sigma/\partial n_-)\,\mathrm{d}s$, where $\mathrm{d}s$ is the arclength on $\mathrm{supp}(\sigma) = \{|z| = 1\} \cup (\bigcup_{k=1}^n \{|z-\alpha_k| = r\})$, and $n_\pm$ are the inner and the outer normals. We now use Green's identity $\iint_G u\Delta v\,\mathrm{d}A = \int_{\partial G} u\,\frac{\partial v}{\partial n}\,\mathrm{d}s - \iint_G \nabla u\cdot\nabla v\,\mathrm{d}A$ with $u = \phi$ and $v = p_\sigma$ in each component G of $\{|z| < R\}\setminus\mathrm{supp}(\sigma)$. Since $\Delta p_\sigma = 0$ in G, adding the identities for all G, we obtain that

$$\left|\int \phi\,\mathrm{d}\sigma\right| = \frac{1}{2\pi}\left|\iint\limits_{|z|\leqslant R} \nabla\phi\cdot\nabla p_\sigma\,\mathrm{d}A\right| \leqslant \frac{1}{2\pi}\sqrt{D[\phi]}\sqrt{D[p_\sigma]},$$

where $D[\phi] = \iint(\phi_x^2 + \phi_y^2)\,\mathrm{d}A$ is the Dirichlet integral of ϕ. It is known that $D[p_\sigma] = 2\pi I[\sigma]$ [7, Thm 1.20], where $I[\sigma] = -\iint \log|z-t|\,\mathrm{d}\sigma(z)\,\mathrm{d}\sigma(t) = \int p_\sigma\,\mathrm{d}\sigma$. Since $p_\mu(z) = -\log\max(1,|z|)$, we observe that $\int p_\mu\,\mathrm{d}\mu = 0$, so that $I[\sigma] = \int p_{\tau_n^r}\,\mathrm{d}\tau_n^r - 2\int p_\mu\,\mathrm{d}\tau_n^r$. Further, $-\int p_\mu\,\mathrm{d}\tau_n^r = \int \log\max(1,|z|)\,\mathrm{d}\tau_n^r(z) \leqslant (\sum_{|\alpha_k|\leqslant 1+r}\log(1+2r) + \sum_{|\alpha_k|>1+r}\log|\alpha_k|)/n \leqslant \log(1+2r) + \frac{1}{n}\log M(P_n) - \frac{1}{n}\log|a_n|$. We also have that $\int p_{\tau_n^r}\,\mathrm{d}\tau_n^r \leqslant (-\sum_{j\neq k}\log|\alpha_j - \alpha_k| - n\log r)/n^2$. We next combine the energy estimates to obtain

$$I[\sigma] \leqslant \frac{2}{n}\log M(P_n) - \frac{1}{n^2}\log\left|a_n^2\Delta(P_n)\right| - \frac{1}{n}\log r + 4r.$$

Collecting all estimates, we proceed with $|\int\phi\,\mathrm{d}\tau_n - \int\phi\,\mathrm{d}\mu| \leqslant |\int\phi\,\mathrm{d}\tau_n - \int\phi\,\mathrm{d}\tau_n^r| + |\int\phi\,\mathrm{d}\tau_n^r - \int\phi\,\mathrm{d}\mu| \leqslant \omega_\phi(r) + \sqrt{D[\phi]}\sqrt{D[p_\sigma]}/(2\pi) = \omega_\phi(r) + \sqrt{D[\phi]}\sqrt{I[\sigma]/(2\pi)}$. Thus we arrive at the main inequality:

$$\left|\int\phi\,\mathrm{d}\tau_n - \int\phi\,\mathrm{d}\mu\right| \leqslant \omega_\phi(r) + \sqrt{\frac{D[\phi]}{2\pi}}\left(\frac{2}{n}\log M(P_n) - \frac{1}{n^2}\log\left|a_n^2\Delta(P_n)\right| - \frac{1}{n}\log r + 4r\right)^{1/2}. \tag{3}$$

Note that $D[\phi] \leqslant 2\pi R^2 A^2$, as $|\phi_x| \leqslant A$ and $|\phi_y| \leqslant A$ a.e. in $\mathbb{C}$. Also, $\omega_\phi(r) \leqslant Ar$. Since $|\Delta(P_n)| \geqslant 1$ and $|a_n| \geqslant 1$, we have $|a_n^2\Delta(P_n)| \geqslant 1$. Hence (2) follows from (3) by letting $r = 1/\max(n, M(P_n))$. □

Proof of Corollary 1.6. Since P_n has real coefficients, we have that $s_n = \int z\,\mathrm{d}\tau_n(z) = \int \Re(z)\,\mathrm{d}\tau_n(z)$. We let $\phi(z) = \Re(z)$, $|z| \leqslant 1$; $\phi(z) = \Re(z)(1 - \log|z|)$, $1 \leqslant |z| \leqslant \mathrm{e}$; and $\phi(z) = 0$, $|z| \geqslant \mathrm{e}$. An elementary computation shows that $|\phi_x(z)| \leqslant 1$ and $|\phi_y(z)| \leqslant 1/2$ for all $z = x + \mathrm{i}y \in \mathbb{C}$. The Mean Value Theorem gives $|\phi(z) - \phi(t)| \leqslant |z-t|\max_{\mathbb{C}}\sqrt{\phi_x^2 + \phi_y^2}$. Hence we can use Theorem 1.5 with $A = \sqrt{5}/2$ and $R = \mathrm{e}$. □

Proof of Corollary 1.7. Note that $\log|P_n(z)| = n\int\log|z-w|\,\mathrm{d}\tau_n(w)$. For $|z| = 1 + 1/n$, we let $\phi(w) = \log|z-w|$, $|w| \leqslant 1$; $\phi(w) = (1 - \log|w|)\log|1 - \bar{z}w|$, $1 \leqslant |w| \leqslant \mathrm{e}$; and $\phi(z) = 0$, $|w| \geqslant \mathrm{e}$. Then $|\phi_x(w)| = \mathrm{O}(|z-w|^{-1})$, $|w| \leqslant 1$; $|\phi_x(w)| = \mathrm{O}(|1-\bar{z}w|^{-1})$, $1 \leqslant |w| \leqslant \mathrm{e}$; and the same estimates hold for $|\phi_y|$. Hence $D[\phi] = \mathrm{O}(\iint_{|w|\leqslant 1}|z-w|^{-2}\,\mathrm{d}A(w)) = \mathrm{O}(\int_{1/n}^1 r^{-1}\,\mathrm{d}r) = \mathrm{O}(\log n)$, and $\omega_\phi(r) \leqslant r\max_{\mathbb{C}}\sqrt{\phi_x^2 + \phi_y^2} = r\mathrm{O}(n)$. Let $r = 1/n^2$ and use (3) to obtain $|\log|P_n(z)| - n\log|z|| = \mathrm{O}(\sqrt{n}\log n)$. □

References

[1] V.V. Andrievskii, H.-P. Blatt, Discrepancy of Signed Measures and Polynomial Approximation, Springer-Verlag, New York, 2002.
[2] Y. Bilu, Limit distribution of small points on algebraic tori, Duke Math. J. 89 (1997) 465–476.
[3] P. Borwein, Computational Excursions in Analysis and Number Theory, Springer-Verlag, New York, 2002.
[4] P. Erdős, P. Turán, On the distribution of roots of polynomials, Ann. Math. 51 (1950) 105–119.
[5] C. Favre, J. Rivera-Letelier, Equidistribution quantitative des points de petite hauteur sur la droite projective, Math. Ann. 335 (2006) 311–361; Corrigendum in Math. Ann. 339 (2007) 799–801.
[6] L. Kronecker, Zwei Sätze über Gleichungen mit ganzzahligen Coëfficienten, J. Reine Angew. Math. 53 (1857) 173–175.
[7] N.S. Landkof, Foundations of Modern Potential Theory, Springer-Verlag, New York, 1972.
[8] E.B. Saff, V. Totik, Logarithmic Potentials with External Fields, Springer-Verlag, Berlin, 1997.
[9] I. Schur, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, Math. Z. 1 (1918) 377–402.