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Functional Analysis

A characterization of upper triangular trace class matrices

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Abstract

As a consequence of the vector-valued Hardy inequality it is given a characterization of upper triangular trace class matrices completely similar to that of classical Hardy space of analytic functions H^1, as may be found for instance in Pavlović's book. ***To cite this article: N. Popa, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une caractérisation de la classe des matrices supérieurement triangulaires à trace. On donne une caractérisation de la classe des matrices supérieurement triangulaires à trace comme une conséquence de l'inégalité vectorielle de Hardy. Cette caractérisation est complètement similaire de celle valable por les espaces de Hardy. ***Pour citer cet article : N. Popa, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

In the book of Pavlović ([3, page 96]) there is the following beautiful characterization of functions belonging to the Hardy space $H^1 = \{f : D \to \mathbb{C},\ \text{such that } f \text{ is analytic and } \|f\|_1 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})\,dt < \infty\}$:

Pavlović's Theorem. *For a function f analytic in D the following assertions are equivalent*:

(a) $f \in H^1$;

(b) $\sup_n \frac{1}{a_n} \sum_{j=0}^{n} \frac{1}{j+1} \|s_j(f)\|_1 < \infty$;

(c) $\sup_n \|P_n f\|_1 < \infty$.

Here, for a function f analytic in D let

$$P_n f = \frac{1}{a_n} \sum_{j=0}^{n} \frac{1}{j+1} s_j(f), \quad \text{where } a_n = \sum_{j=0}^{n} \frac{1}{j+1} \quad (n = 0, 1, 2, \ldots)$$

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and $s_j(f)$ are the partial sums of the Taylor series of f.

An analogue of this result using the following vector-valued Hardy inequality (see [1] for this inequality):

$$\sum_{k\geqslant 0}(k+1)^{-1}\|\hat{f}(k)\|_1 \leqslant C\|f\|_1 \quad \text{for } f\in H_X^1 \tag{1}$$

is also true and is presented below. Here, as in [2], X is a complex Banach space, L_X^1 is the space of all X-valued 2π-periodic functions on the real line $\mathbb{R}$ which are Bochner absolutely integrable under the norm

$$\|f\|_1 = \left[(2\pi)^{-1}\int_{-\pi}^{\pi}\|f(t)\|\,\mathrm{d}t\right]^{1/p}, \qquad H_X^1 = \{f\in L_X^1;\ \hat{f}(j)=0 \text{ for } j<0\},$$

where $\hat{f}(j) = (2\pi)^{-1}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}jt}f(t)\,\mathrm{d}t$.

We explain some notations and notions used in what follows.

T_1 means the space of all upper triangular matrices of trace class, endowed with the usual trace class norm $\|A\| = \sum_{n=1}^{\infty}\alpha_n(A)$, where $\alpha_n(A)$ is the nth-singular number of A, i.e. the nth-eigenvalue of the $(AA^*)^{1/2}$.

We use the *Schur* (*Hadamard*) *product* $A*B$ of two matrices A and B as being the matrix C whose entries are defined by $c_{i,j} = a_{i,j}b_{i,j}$ for all indices i and j.

A special class of infinite matrices which is used often in this note, is the class of *Toeplitz matrices*.

Let $A = (a_{i,j})_{i,j\geqslant 1}$ be an infinite matrix. If there is a sequence of complex numbers $(a_k)_{k=-\infty}^{+\infty}$, such that $a_{i,j} = a_{j-i}$ for all $i,j\in\mathbb{N}$, then A is called *a Toeplitz matrix*. To a Toeplitz matrix A given by the sequence $(a_k)_{k\in\mathbb{Z}}$ we associate a 2π-periodic distribution $f = \sum_{k=0}^{\infty}a_k\mathrm{e}^{\mathrm{i}kt}$, where $t\in[0,1)$ and conversely.

Now we have the following result:

Theorem 1. *Let A be an upper triangular matrix. The following assertions are equivalent*:

(a) $A\in T_1$;

(b) $\sup_n \frac{1}{a_n}\sum_{j=0}^{n}\frac{1}{j+1}\|s_j(A)\| < \infty$;

(c) $\sup_n \|P_nA\| < \infty$.

Here

$$P_nA = \frac{1}{a_n}\sum_{j=0}^{n}\frac{1}{j+1}s_j(A), \quad \text{where } a_n = \sum_{j=0}^{n}\frac{1}{j+1} \quad (n=0,1,2,\ldots),$$

$s_j(A) = \sum_{k=0}^{j}A_k$ and A_k is the kth-*diagonal matrix* of A, i.e. A_k is the matrix whose entries $a'_{i,j}$ are given by

$$a'_{i,j} = \begin{cases} a_{i,j} & \text{if } j-i=k,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Obviously (b) ⇒ (c).

(a) ⇒ (b). Let $A\in T_1$, and for fixed $n\geqslant 2$, $w\in D$, and $r = 1-\frac{1}{n}<1$, define the matrix-valued function $g(z) = (1-rz)^{-1}[A*C(rwz)]$ $(|z|\leqslant 1)$, where $C(z)$ is the Toeplitz matrix corresponding to the function $\frac{1}{1-z}$ for each $z\in D$.

Then we have:

$$\begin{aligned} g(z) &= \left(\sum_{k=0}^{\infty}A_kr^kw^kz^k\right)\left(\sum_{l=0}^{\infty}r^lz^l\right) = \sum_{k,l=0}^{\infty}A_kw^kr^{k+l}z^{k+l}\\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}A_kw^k\right)r^mz^m = \sum_{m=0}^{\infty}s_m\big(A*C(w)\big)r^mz^m. \end{aligned}$$

Hence $\hat{g}(m) = s_m(A*C(w))r^mz^m$, $m=0,1,2,\ldots$.

It is well known (and easy to see) that

$$\|s_m A\|_{T_1} \leqslant C \ln(m+1) \|A\|_{T_1} \quad \forall A \in T_1 \text{ and } m \in \mathbb{N}, \tag{2}$$

where $C > 0$ is an absolute constant.

$g \in H^1_{T_1}$ since, by (2), we have

$$\left\|s_m\big(A * C(w)\big)\right\|_{T_1} \leqslant \frac{1}{1-|w|}\|s_m A\|_{T_1} \leqslant \frac{C\ln(m+1)}{1-|w|} \quad \forall m \in \mathbb{N} \text{ and } |w|<1$$

therefore

$$\sum_{m=0}^{\infty} \left\|s_m\big(A * C(w)\big)\right\|_{T_1} r^m \leqslant \frac{C\sum_{m=0}^{\infty} r^m \ln(m+1)}{1-|w|} < \infty.$$

Then

$$\sum_{j=0}^{\infty} \frac{1}{j+1}\left\|s_j\big(A * C(w)\big)\right\|_{T_1} r^j = \sum_{j=0}^{\infty} \frac{1}{j+1}\left\|\hat{g}(j)\right\|_{T_1} \quad \big(\text{by (1) for } X = T_1\big)$$

$$\leqslant C\|g\|_{H^1_{T_1}} = \frac{\|A * C(rw\mathrm{e}^{\mathrm{i}t})\|_{T_1}}{|1-r\mathrm{e}^{\mathrm{i}t}|} \quad \text{for all } t \in [0, 2\pi).$$

Since $r^j = (1-\frac{1}{n})^j \geqslant c \ \forall 0 \leqslant j \leqslant n$, where $c > 0$ is an absolute constant, we have:

$$\sum_{j=0}^{n} \frac{1}{j+1}\left\|s_j\big(A * C(w)\big)\right\|_{T_1} \leqslant C\int_0^{2\pi} \left\|g\big(r\mathrm{e}^{\mathrm{i}t}\big)\right\|_{T_1} \frac{dt}{2\pi} = C\int_0^{2\pi} \frac{\|A * C(rw\mathrm{e}^{\mathrm{i}t})\|_{T_1}}{|1-r\mathrm{e}^{\mathrm{i}t}|}\frac{\mathrm{d}t}{2\pi}.$$

Integrating this inequality over the circle $|w| = 1$ and since $s_j(A * C(w)) = s_j(A) * C(w)$, we find, using $\lim_{w\to e^{i\theta}} \|s_j(A) * C(w)\|_{T_1} = \|s_j A * C(e^{i\theta})\|_{T_1} \ \forall j$, that

$$\sum_{j=0}^{n} \frac{1}{j+1}\int_0^{2\pi} \left\|s_j A * C\big(\mathrm{e}^{\mathrm{i}\theta}\big)\right\|_{T_1}\frac{\mathrm{d}\theta}{2\pi} \leqslant C' \int_0^{2\pi}\int_0^{2\pi} \frac{\|A * C(r\mathrm{e}^{\mathrm{i}(\theta+t)})\|_{T_1}}{|1-r\mathrm{e}^{\mathrm{i}t}|}\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi}$$

$$= (\text{by Fubini's theorem})\ C' \int_0^{2\pi}\left(\int_0^{2\pi} \left\|A * P_r(t+\theta)\right\|_{T_1}\frac{\mathrm{d}\theta}{2\pi}\right)\frac{\mathrm{d}t}{2\pi|1-r\mathrm{e}^{\mathrm{i}t}|} \leqslant C''\|A\|_{T_1}\ln n,$$

where $P_r(t+\theta)$ is the usual Poisson kernel on the unit circle and $C'' > 0$ is an absolute constant.

But denoting by E_θ the Toeplitz matrix corresponding to δ_θ the Dirac measure concentrated in θ, it is easy to see that

$$\|B\| = \|B * E_\theta\|. \tag{$*$}$$

We have obviously that

$$\frac{1}{2\pi}\int_0^{2\pi} \left\|s_j(A) * C\big(\mathrm{e}^{\mathrm{i}\theta}\big)\right\| \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \left\|s_j(A) * E_\theta\right\| \mathrm{d}\theta$$

and by $(*)$ it follows that:

$$\sum_{j=0}^{n} \frac{1}{j+1}\left\|s_j(A)\right\| \leqslant \sum_{j=1}^{n} \frac{1}{j+1}\int_0^{2\pi} \left\|s_j(A) * C\big(\mathrm{e}^{\mathrm{i}\theta}\big)\right\|\frac{\mathrm{d}\theta}{2\pi} \leqslant C\|A\|\ln n,$$

that is $\frac{1}{a_n}\sum_{j=0}^{n}\frac{1}{j+1}\|s_j(A)\| \leqslant C_1\|A\|$ and (b) holds.

(c) ⇒ (a). First, it is clear that if A is a finite matrix, then $\|A\|_{S_1} \leqslant \sup_n \|P_n A\|_{S_1}$. Now assume that A is any matrix such that $\sup_n \|P_n A\|_{S_1} < \infty$. Let E_m be the canonical projection which projects a matrix to its submatrix of order m at the left upper corner. Since P_n and E_m commute, we find that $\sup_m \sup_n \|P_n E_m A\|_{S_1} < \infty$.

By the preceding remark, we have $\sup_m \|E_m A\|_{S_1} \leqslant \sup_n \|P_n E_m A\|_{S_1}$; whence $A \in S_1$ and $\|A\|_{S_1} \leqslant \sup_n \|P_n A\|_{S_1}$. This inequality holds without the assumption that A is upper triangular. □

A simple consequence of the previous theorem is:

Corollary 2. *If $A \in T_1$, then*

$$\lim_n \frac{1}{a_n} \sum_{j=0}^{n} \frac{1}{j+1} \left\| A - s_j(A) \right\| = 0 \tag{3}$$

and, consequently,

$$\lim_n \frac{1}{a_n} \sum_{j=0}^{n} \frac{1}{j+1} \left\| s_j(A) \right\| = \|A\|. \tag{4}$$

Proof. Obviously (3) holds if A is a finite matrix. Since finite matrices are dense in T_1 the proof of (3) is over. The second assertion follows immediately from (3). □

We remark that B. Smith [4] proved 1983 the relation (4) for $f \in H^1$ instead of $A \in T_1$, what motived Pavlović to give his theorem.

As a consequence of this result we have:

Corollary 3. *If $A \in T_1$ then* $\liminf_{n\to\infty} \|A - s_n(A)\| = 0$.

Remark 4. 1. A Banach space X is of $(H^1 - \ell^1)$-Fourier type provided for every multiplier sequence $m = (m_k)_{k\geqslant 0}$ such that there exists a constant $K = K(m, X)$ so that for every analytic trigonometric polynomial f $(\sum_{j=0}^{\infty} |m_j \hat{f}(n_j)|) \leqslant K\|f\|$, we have the same inequality where the norm $\|\cdot\|_X$ is used instead of the absolute value $|\cdot|$. It was proved in [1] that S_1 has the $(H^1 - \ell^1)$-Fourier type. Then the following matrix version of Hardy's inequality of [2] holds:

Generalized Shield's inequality. *There is a constant $C > 1$ such that given any set $n_1 < n_2 < \cdots < n_k \subset \mathbb{Z}$, and $A = \sum_{k=1}^{\infty} A_{n_k} \in S_1$, we have $\sum_{k=1}^{\infty} \frac{\|A_{n_k}\|_{S_1}}{k} \leqslant C\|A\|_{S_1}$.*

Indeed, view the $(H^1 - \ell^1)$-Fourier type property of S_1, the inequality above holds for every upper triangular matrix A. Denoting by $\mathcal{S}$ the unilateral shift to the right, it is easy to see that $\mathcal{S}^n$, is a bounded operator on S_1 for some fixed $n \in \mathbb{N}$. (Of course the norm of $\mathcal{S}^n$ may depend on n.) But $\mathcal{S}^{n_1} A$ is an upper triangular matrix, so the generalized Shield's inequality holds.

2. From the above inequality it follows also the matrix version of the positive answer to a Littlewood conjecture (see [2]).

There is a constant $C > 1$ such that given any set $\{n_1 < n_2 < \cdots < n_N\} \subset \mathbb{Z}$ and a matrix $A = \sum_{k=1}^{N} A_{n_k}$ with $\|A_{n_k}\|_{S_1} \geqslant 1$ for all k, then $\|A\|_{S_1} \geqslant C \log N$.

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