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Probability Theory

Finite time extinction for solutions to fast diffusion stochastic porous media equations

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Abstract

We prove that the solutions to fast diffusion stochastic porous media equations have finite time extinction with strictly positive probability. ***To cite this article: V. Barbu et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Extinction en temps fini pour les solutions des équations des milieu poreux avec diffusion rapide. Nous prouvons l'extinction avec une probabilité strictement positive pour les solutions des équations des milieux poreux avec diffusion rapide. ***Pour citer cet article : V. Barbu et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Consider the stochastic porous media equation

$$\begin{cases} dX(t) - \rho\Delta(|X|^\alpha(t)\,\mathrm{sign}\,X(t))\,dt - \Delta(\tilde{\Psi}(X(t))\,dt = \sigma(X(t))\,dW(t), & \text{in } (0,\infty)\times\mathcal{O}, \\ X = 0 \quad \text{on } (0,\infty)\times\partial\mathcal{O}, \quad X(0,x) = x \quad \text{on } \mathcal{O}, \end{cases} \tag{1}$$

where $\rho > 0$, $\alpha \in (0,1)$, $\tilde{\Psi}$ is a continuous monotonically nondecreasing function of linear growth and $\sigma(X)\,dW = \sum_{k=1}^\infty \mu_k X e_k\,d\beta_k$, $t \geqslant 0$, where $\{\beta_k\}$ is a sequence of independent real Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and $\{e_k\}$ is an orthonormal basis in $L^2(\mathcal{O})$ which for convenience will be taken as the eigenfunction system for the Laplace operator with Dirichlet boundary conditions, i.e., $-\Delta e_k = \lambda_k e_k$ in $\mathcal{O}$, $e_k = 0$ on $\partial\mathcal{O}$, where $\mathcal{O}$ is an open and bounded subset of $\mathbb{R}^d$, with smooth boundary $\partial\mathcal{O}$. We shall assume that $\sum_{k=1}^\infty \mu_k^2\lambda_k^2 < \infty$. Eq. (1) for $0 < \alpha < 1$ is relevant in the mathematical modelling of the dynamics of an ideal gas in

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a porous medium and, in particular, in a plasma fast diffusion model (for $\alpha = 1/2$) (see e.g. [4]). The existence and uniqueness of a strong solution in the sense to be defined below was studied in [1–3,5] for more general nonlinear stochastic equations of the form (1). In [3] (see also [1]) it was also proven that for $\alpha = 0$ and $d = 1$ the solution $X = X(t,x)$ to (1) has the finite extinction property: $\mathbb{P}(\tau \leqslant n) \geqslant 1 - \frac{|x|_{-1}}{\rho\gamma}(\int_0^n \mathrm{e}^{-C_N s}\,\mathrm{d}s)^{-1}$ for $|x|_{-1} < C_N^{-1}\rho\gamma$ where $\tau = \inf\{t \geqslant 0\colon\ |X(t,x)|_{-1} = 0\} = \sup\{t \geqslant 0\colon\ |X(t,x)|_{-1} > 0\}$ and C_N, γ are constants related to the Wiener process W and respectively to the domain $\mathcal{O} \subset \mathbb{R}^1$.

The following notations will be used in the sequel. $H = L^2(\mathcal{O})$, $p \geqslant 1$, with the norm denoted by $|\cdot|_2$ and scalar product $\langle\cdot,\cdot\rangle$. $H^{-1}(\mathcal{O})$ is the dual of the Sobolev space $H_0^1(\mathcal{O})$ and is endowed with the scalar product $\langle u, v\rangle_{-1} = \langle u, (-\Delta)^{-1}v\rangle$, where Δ is the Laplace operator with domain $H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. All processes $X = X(t)$ arising here are adapted with respect to the filtration $\{\mathcal{F}_t\}$. For a Banach space E, $L_W^p(0,T;E)$ denotes the space of all adapted processes in $L^p(0,T;E)$. We shall use standard notation for Sobolev spaces and spaces of integrable functions on $\mathcal{O}$.

2. The main result

Definition 2.1. Let $x \in H$. An H-valued continuous $(\mathcal{F}_t)$-adapted process $X = X(t,x)$ is called a solution to (1) on $[0,T]$ if $X \in L^p(\Omega \times (0,T) \times \mathcal{O}) \cap L^2(0,T;L^2(\Omega,H))$, $p \geqslant 2$, such that $\mathbb{P}$-a.s. $\forall j \in \mathbb{N}$, $t \in [0,T]$,

$$\begin{aligned}\langle X(t,x), e_j\rangle = \langle x, e_j\rangle &+ \int_0^t\int_{\mathcal{O}} \big(\rho\big|X(s,x)(\xi)\big|^{\alpha}\operatorname{sign} X(s,x)(\xi) + \tilde{\Psi}\big(X(s,x)(\xi)\big)\big)\Delta e_j(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\ &+ \sum_{k=1}^{\infty}\mu_k\int_0^t \langle X(s,x)e_k, e_j\rangle\,\mathrm{d}\beta_k(s). \end{aligned} \tag{2}$$

For $x \in L^p(\mathcal{O})$, $p \geqslant 4$ and $d = 1,2,3$ there is a unique solution $X \in L_W^\infty(0,T;L^p(\Omega,H))$ to (1) in the sense of Definition 2.1. Moreover, if $x \geqslant 0$ a.e. in $\mathcal{O}$ then $X \geqslant 0$ a.e. in $\Omega \times [0,T] \times \mathcal{O}$.

By the proof of [3, Theorem 2.2] and [3, Proposition 3.4] we also know that for $\lambda \to 0$,

$$\begin{cases} X_\lambda \to X \text{ strongly both in } L^2(0,T;L^2(\Omega,L^2(\mathcal{O}))) \text{ and in } L^2(\Omega;C([0,T];H)),\\ \text{weakly in } L^p(\Omega\times(0,T)\times\mathcal{O}), \text{ and weak* in } L^\infty(0,T;L^p(\Omega;L^p(\mathcal{O}))),\end{cases} \tag{3}$$

where X_λ, $\lambda > 0$, is the solution to approximating equation

$$\begin{cases} \mathrm{d}X_\lambda(t) - \Delta(\Psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t) + \tilde{\Psi}(X_\lambda(t)))\,\mathrm{d}t = \sigma(X_\lambda(t))\,\mathrm{d}W(t),\\ \Psi_\lambda(X_\lambda) + \lambda X_\lambda + \tilde{\Psi}(X_\lambda) = 0 \quad \text{on } \partial\mathcal{O}, \quad X_\lambda(0,x) = x,\\ \Psi_\lambda(x) = \frac{1}{\lambda}(x - (1+\lambda\Psi_0)^{-1}(x)) = \Psi_0((1+\lambda\Psi_0)^{-1}(x)), \quad \Psi_0(x) = \rho|x|^\alpha\operatorname{sign} x.\end{cases} \tag{4}$$

Everywhere in the sequel $X = X(t,x)$ is the solution to (1) in the sense of Definition 2.1 where $x \in L^4(\mathcal{O})$. Below γ shall denote the minimal constant arising in the Sobolev embedding $L^{\alpha+1}(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ (see (7) below) and $C^* = \sum_{k=1}^\infty \mu_k^2|e_k|^2_{H_0^1(\mathcal{O})} = \sum_{k=1}^\infty \mu_k^2\lambda_k^2$. Theorem 2.2 is the main result of the paper.

Theorem 2.2. *Assume that $d = 1,2,3$ and that $0 < \alpha < 1$ if $d = 1,2$, $\frac{1}{5} \leqslant \alpha < 1$ if $d = 3$. Let $\tau := \inf\{t \geqslant 0\colon |X(t,x)|_{-1} = 0\}$. Then we have $|X(t,x)|_{-1} = 0$, for $t \geqslant \tau$, $\mathbb{P}$-a.s. Furthermore*

$$\mathbb{P}(\tau \leqslant t) \geqslant 1 - \frac{|x|_{-1}^{1-\alpha}}{(1-\alpha)\rho\gamma^{1+\alpha}}\left(\int_0^t \mathrm{e}^{-C^*(1-\alpha)s}\,\mathrm{d}s\right)^{-1}.$$

In particular, if $|x|_{-1}^{1-\alpha} < \rho\gamma^{1+\alpha}/C^*$, then $\mathbb{P}(\tau < \infty) > 0$, and if $C^* = 0$, then $\tau \leqslant |x|_{-1}^{1-\alpha}/((1-\alpha)\rho\gamma^{1+\alpha})$.

Remark 1. This result extends to $\mathcal{O} \subset \mathbb{R}^d$ with $d \geqslant 4$, if $\alpha \in [\frac{d-2}{d+2}, 1)$. However, we have to strengthen the assumption on μ_k, $k \in \mathbb{N}$, see [1, Section 4] and in particular [6, Remark 2.9(iii)] for a detailed discussion.

3. Proof of Theorem 2.2

We shall proceed as in the proof of [3, Theorem 4.2]. Consider the solution $X_\lambda \in L^2_W(0,T;L^2(\Omega;H^1_0(\mathcal{O})))$ to Eq. (4). Then by applying the classical Itô formula to the real valued semi-martingale $|X_\lambda(t)|^2_{-1}$, $t\in[0,T]$, and to the function $\varphi_\varepsilon(r)=(r+\varepsilon^2)^{(1-\alpha)/2}$, $r\in\mathbb{R}$, we find that

$$\begin{aligned}
&d\varphi_\varepsilon\big(\big|X_\lambda(t)\big|^2_{-1}\big)+(1-\alpha)\big(\big|X_\lambda(t)\big|^2_{-1}+\varepsilon^2\big)^{-(1+\alpha)/2}\big\langle X_\lambda(t),\Psi_\lambda\big(X_\lambda(t)\big)+\lambda X_\lambda(t)+\tilde\Psi_\lambda\big(X_\lambda(t)\big)\big\rangle\,dt\\
&=\frac12\sum_{k=1}^\infty\mu_k^2(1-\alpha)\frac{|X_\lambda(t)e_k|^2_{-1}(|X_\lambda(t)|^2_{-1}+\varepsilon^2)-(1-\alpha)^2|\langle X_\lambda(t)e_k,X_\lambda(t)\rangle_{-1}|^2)}{(|X_\lambda(t)|^2_{-1}+\varepsilon^2)^{(3+\alpha)/2}}\,dt\\
&\quad+\big\langle\sigma\big(X_\lambda(t)\big)\,dW(t),\varphi'_\varepsilon\big(\big|X_\lambda(t)\big|^2_{-1}\big)X_\lambda(t)\big\rangle_{-1}\\
&\leqslant\frac12\sum_{k=1}^\infty\mu_k^2\frac{(1-\alpha)|X_\lambda(t)e_k|^2_{-1}}{(|X_\lambda(t)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,dt+\big\langle\sigma\big(X_\lambda(t)\big)\,dW(t),\varphi'_\varepsilon\big(\big|X_\lambda(t)\big|^2_{-1}\big)X_\lambda(t)\big\rangle_{-1}\\
&\leqslant C^*\frac{(1-\alpha)|X_\lambda(t)e_k|^2_{-1}}{(|X_\lambda(t)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,dt+\big\langle\sigma\big(X_\lambda(t)\big)\,dW(t),\varphi'_\varepsilon\big(\big|X_\lambda(t)\big|^2_{-1}\big)X_\lambda(t)\big\rangle_{-1}.
\end{aligned}\tag{5}$$

Then letting $\lambda\to0$, by (3) we get that $\liminf_{\lambda\to0}\int_0^T\langle\Psi_\lambda(X_\lambda(t)),X_\lambda(t)\rangle\,dt\geqslant\rho\int_0^T|X(t)|^{1+\alpha}_{L^{1+\alpha}(\mathcal{O})}\,dt$, $\mathbb{P}$-a.s. and hence

$$\begin{aligned}
&\varphi_\varepsilon\big(\big|X(t)\big|^2_{-1}\big)+(1-\alpha)\rho\int_r^t\frac{|X(s)|^{\alpha+1}_{L^{\alpha+1}(\mathcal{O})}}{(|X(s)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,ds\leqslant\varphi_\varepsilon\big(\big|X(r)\big|^2_{-1}\big)\\
&\quad+C^*\int_r^t\frac{(1-\alpha)|X(s)|^2_{-1}}{(|X(s)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,ds+2\int_r^t\big\langle\sigma\big(X(s)\big)\,dW(s),\varphi'_\varepsilon\big(\big|X(s)\big|^2_{-1}\big)X(s)\big\rangle_{-1},\quad\mathbb{P}\text{-a.s., } r<t.
\end{aligned}\tag{6}$$

Next by the Sobolev embedding theorem we have

$$|u|_{-1}\leqslant\gamma|u|_{L^{\alpha+1}(\mathcal{O})},\quad\forall u\in L^{\alpha+1}(\mathcal{O}),\quad\text{if } d>2 \text{ and } \alpha\geqslant\tfrac{d-2}{d+2},\text{ and }\forall\alpha>0,\text{ if } d=1,2.\tag{7}$$

Then substituting (7) into (6) we get

$$\begin{aligned}
&\varphi_\varepsilon\big(\big|X(t)\big|^2_{-1}\big)+(1-\alpha)\rho\gamma^{1+\alpha}\int_r^t\frac{|X(s)|^{\alpha+1}_{-1}}{(|X(s)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,ds\leqslant\varphi_\varepsilon\big(\big|X(r)\big|^2_{-1}\big)\\
&\quad+C^*\int_r^t\frac{(1-\alpha)|X(s)|^2_{-1}}{(|X(s)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,ds+\int_r^t\big\langle\sigma\big(X(s)\big)\,dW(s),\varphi'_\varepsilon\big(\big|X(s)\big|^2_{-1}\big)X(s)\big\rangle_{-1},\quad\mathbb{P}\text{-a.s., } r<t.
\end{aligned}\tag{8}$$

Now for $\epsilon\to0$ we have

$$\begin{aligned}
&\big|X(t)\big|^{1-\alpha}_{-1}+(1-\alpha)\rho\gamma^{1+\alpha}\int_r^t 1_{\{|X(s)|_{-1}>0\}}\,ds\leqslant\big|X(r)\big|^{1-\alpha}_{-1}+C^*(1-\alpha)\int_r^t\big|X(s)\big|^{1-\alpha}_{-1}\,ds\\
&\quad+(1-\alpha)\int_r^t\big\langle\sigma\big(X(s)\big)\,dW(s),\big|X(s)\big|^{-(\alpha+1)}_{-1}X(s)\big\rangle_{-1},\quad\mathbb{P}\text{-a.s., } r<t.
\end{aligned}$$

Hence by Itô's product rule

$$\begin{aligned}
&e^{-C^*(1-\alpha)t}\big|X(t)\big|^{1-\alpha}_{-1}+(1-\alpha)\rho\gamma^{1+\alpha}\int_r^t e^{-C^*(1-\alpha)s}1_{\{|X(s)|_{-1}>0\}}\,ds\\
&\quad\leqslant e^{-C^*(1-\alpha)r}\big|X(r)\big|^{1-\alpha}_{-1}+(1-\alpha)\int_r^t e^{-C^*(1-\alpha)s}\big\langle\sigma\big(X(s)\big)\,dW(s),\big|X(s)\big|^{-(\alpha+1)}_{-1}X(s)\big\rangle_{-1},\quad\mathbb{P}\text{-a.s., } r<t.
\end{aligned}\tag{9}$$

From this it immediately follows that $e^{-C^*(1-\alpha)t}|X(t)|_{-1}^{1-\alpha}$, $t \geqslant 0$, is an $(\mathcal{F}_t)$-supermartingale, hence $|X(t)|_{-1} = 0$ for all $t \geqslant \tau$. So, (9) with $r = 0$ after taking expectation implies that $\int_0^t e^{-C^*(1-\alpha)s}\mathbb{P}(\tau > s)\,ds \leqslant |x|_{-1}^{1-\alpha}/((1-\alpha)\rho\gamma^{1+\alpha})$, $t \geqslant 0$. This implies that $\mathbb{P}(\tau > t) \leqslant |x|_{-1}^{1-\alpha}/((1-\alpha)\rho\gamma^{1+\alpha})(\int_0^t e^{-C^*(1-\alpha)s}\,ds)^{-1}$, $t \geqslant 0$, and the assertion follows.

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