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Functional Analysis

A direct proof of the functional Santaló inequality

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Abstract

We give a simple proof of a functional version of the Blaschke–Santaló inequality due to Artstein, Klartag and Milman. The proof is by induction on the dimension and does not use the Blaschke–Santaló inequality. ***To cite this article: J. Lehec, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une preuve directe de l'inégalité de Santaló fonctionnelle. On présente une démonstration simple d'une version fonctionnelle de l'inégalité de Blaschke–Santaló, due à Artstein, Klartag et Milman. On procède par récurrence sur la dimension, sans faire appel à l'inégalité ensembliste. ***Pour citer cet article : J. Lehec, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

For $x, y \in \mathbb{R}^n$, we denote their inner product by $\langle x, y\rangle$ and the Euclidean norm of x by $|x|$. If A is a subset of $\mathbb{R}^n$, we let $A^\circ = \{x \in \mathbb{R}^n \mid \forall y \in A, \langle x, y\rangle \leqslant 1\}$ be its polar body. The Blaschke–Santaló inequality states that any convex body K in $\mathbb{R}^n$ with center of mass at 0 satisfies

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n\big(K^\circ\big) \leqslant \mathrm{vol}_n(D)\,\mathrm{vol}_n\big(D^\circ\big) = v_n^2, \tag{1}$$

where vol_n stands for the volume, D for the Euclidean ball and v_n for its volume. Let g be a non-negative Borel function on $\mathbb{R}^n$ satisfying $0 < \int g < \infty$ and $\int |x| g(x)\,\mathrm{d}x < \infty$, then $\mathrm{bar}(g) = (\int g)^{-1}(\int g(x)x\,\mathrm{d}x)$ denotes its center of mass (or barycenter). The center of mass (or centroid) of a measurable subset of $\mathbb{R}^n$ is by definition the barycenter of its indicator function.

Let us state a functional form of (1) due to Artstein, Klartag and Milman [1]. If f is a non-negative Borel function on $\mathbb{R}^n$, the polar function of f is the log-concave function defined by

$$f^\circ(x) = \inf_{y\in\mathbb{R}^n}\big(\mathrm{e}^{-\langle x,y\rangle} f(y)^{-1}\big).$$

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Theorem 1.1 *(Artstein, Klartag, Milman). If f is a non-negative integrable function on $\mathbb{R}^n$ such that f° has its barycenter at* 0*, then*

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \int_{\mathbb{R}^n} f^\circ(y)\,\mathrm{d}y \leqslant \left(\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}|x|^2}\,\mathrm{d}x\right)^2 = (2\pi)^n.$$

In the special case where the function f is even, this result follows from an earlier inequality of Keith Ball [2]; and in [4], Fradelizi and Meyer prove something more general (see also [5]). In the present Note we prove the following:

Theorem 1.2. *Let f and g be non-negative Borel functions on $\mathbb{R}^n$ satisfying the duality relation*

$$\forall x, y \in \mathbb{R}^n, \qquad f(x)g(y) \leqslant \mathrm{e}^{-\langle x,y\rangle}. \tag{2}$$

If f (or g) has its barycenter at 0 *then*

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \int_{\mathbb{R}^n} g(y)\,\mathrm{d}y \leqslant (2\pi)^n. \tag{3}$$

This is slightly stronger than Theorem 1.1 in which the function that has its barycenter at 0 should be log-concave. The point of this Note is not really this improvement, but rather to present a simple proof of Theorem 1.1. Theorem 1.2 yields an improved Blaschke–Santaló inequality, obtained by Lutwak in [6], with a completely different approach.

Corollary 1.3. *Let S be a star-shaped* (*with respect to* 0) *body in $\mathbb{R}^n$ having its centroid at* 0*. Then*

$$\mathrm{vol}_n(S)\,\mathrm{vol}_n\big(S^\circ\big) \leqslant v_n^2. \tag{4}$$

Proof. Let $N_S(x) = \inf\{r > 0 \mid x \in rS\}$ be the gauge of S and $\phi_S = \exp(-\frac{1}{2}N_S^2)$. Integrating ϕ_S and the indicator function of S on level sets of N_S, it is easy to see that $\int_{\mathbb{R}^n} \phi_S = c_n \,\mathrm{vol}_n(S)$ for some constant c_n depending only on the dimension. Replacing S by the Euclidean ball in this equality yields $c_n = (2\pi)^{n/2} v_n^{-1}$. Therefore it is enough to prove that

$$\int \phi_S \int \phi_{S^\circ} \leqslant (2\pi)^n. \tag{5}$$

Similarly, it is easy to see that $\mathrm{bar}(\phi_S) = c_n' \,\mathrm{bar}(S) = 0$. Besides, we have $\langle x, y\rangle \leqslant N_S(x)N_{S^\circ}(y) \leqslant \frac{1}{2}N_S(x)^2 + \frac{1}{2}N_{S^\circ}(y)^2$, for all $x, y \in \mathbb{R}^n$. Thus ϕ_S and ϕ_{S° satisfy (2), then by Theorem 1.2 we get (5). □

2. Main results

Theorem 2.1. *Let f be a non-negative Borel function on $\mathbb{R}^n$ having a barycenter. Let H be an affine hyperplane splitting $\mathbb{R}^n$ into two half-spaces H_+ and H_-. Define $\lambda \in [0, 1]$ by $\lambda \int_{\mathbb{R}^n} f = \int_{H_+} f$. Then there exists $z \in \mathbb{R}^n$ such that for every non-negative Borel function g*

$$\textit{If}\ \big(\forall x, y \in \mathbb{R}^n,\ f(z+x)g(y) \leqslant \mathrm{e}^{-\langle x,y\rangle}\big) \quad \textit{then} \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leqslant \frac{1}{4\lambda(1-\lambda)}(2\pi)^n. \tag{6}$$

In particular, in every median H ($\lambda = \frac{1}{2}$) there is a point z such that for all g

$$\textit{If}\ \big(\forall x, y \in \mathbb{R}^n,\ f(z+x)g(y) \leqslant \mathrm{e}^{-\langle x,y\rangle}\big) \quad \textit{then} \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leqslant (2\pi)^n. \tag{7}$$

A similar result concerning convex bodies (instead of functions) was obtained by Meyer and Pajor in [7].

Let us derive Theorem 1.2 from the latter. Let f, g satisfy (2). Assume for example that $\mathrm{bar}(g) = 0$, then 0 cannot be separated from the support of g by a hyperplane, so there exists $x_1, \ldots, x_{n+1} \in \mathbb{R}^n$ such that 0 belongs to the interior of $\operatorname{conv}\{x_1 \dots x_{n+1}\}$ and $g(x_i) > 0$ for $i = 1 \dots n+1$. Then (2) implies that $f(x) \leqslant C\mathrm{e}^{-\|x\|}$, for some $C > 0$, where $\|x\| = \max(\langle x, x_i\rangle \mid i \leqslant n+1)$. Assume also that $\int f > 0$, then f has a barycenter. Apply the "$\lambda = 1/2$" part of Theorem 2.1 to f. There exists $z \in \mathbb{R}^n$ such that (7) holds. On the other hand, by (2)

$$f(z+x)g(y)\mathrm{e}^{\langle y,z\rangle} \leqslant \mathrm{e}^{-\langle z+x,y\rangle}\mathrm{e}^{\langle y,z\rangle} = \mathrm{e}^{-\langle x,y\rangle}$$

for all $x, y \in \mathbb{R}^n$. Therefore

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \int_{\mathbb{R}^n} g(y)\mathrm{e}^{\langle y,z\rangle}\,\mathrm{d}y \leqslant (2\pi)^n. \tag{8}$$

Integrating with respect to $g(y)\,\mathrm{d}y$ the inequality $1 \leqslant \mathrm{e}^{\langle y,z\rangle} - \langle y, z\rangle$ we get

$$\int_{\mathbb{R}^n} g(y)\,\mathrm{d}y \leqslant \int_{\mathbb{R}^n} g(y)\mathrm{e}^{\langle y,z\rangle}\,\mathrm{d}y - \int_{\mathbb{R}^n} \langle y, z\rangle g(y)\,\mathrm{d}y.$$

Since $\operatorname{bar}(g) = 0$, the latter integral is 0 and together with (8) we obtain (3). Observe also that this proof shows that Theorem 2.1 in dimension n implies Theorem 1.2 in dimension n.

In order to prove Theorem 2.1, we need the following logarithmic form of the Prékopa–Leindler inequality. For details on Prékopa–Leindler, we refer to [3].

Lemma 2.2. *Let ϕ_1, ϕ_2 be non-negative Borel functions on $\mathbb{R}_+$. If $\phi_1(s)\phi_2(t) \leqslant \mathrm{e}^{-st}$ for every s, t in $\mathbb{R}_+$, then*

$$\int_{\mathbb{R}_+} \phi_1(s)\,\mathrm{d}s \int_{\mathbb{R}_+} \phi_2(t)\,\mathrm{d}t \leqslant \frac{\pi}{2}. \tag{9}$$

Proof. Let $f(s) = \phi_1(\mathrm{e}^s)\mathrm{e}^s$, $g(t) = \phi_2(\mathrm{e}^t)\mathrm{e}^t$ and $h(r) = \exp(-\mathrm{e}^{2r}/2)\mathrm{e}^r$. For all $s, t \in \mathbb{R}$ we have $\sqrt{f(s)g(t)} \leqslant h(\frac{t+s}{2})$, hence by Prékopa–Leindler $\int_{\mathbb{R}} f \int_{\mathbb{R}} g \leqslant (\int_{\mathbb{R}} h)^2$. By change of variable, this is the same as $\int_{\mathbb{R}_+} \phi_1 \int_{\mathbb{R}_+} \phi_2 \leqslant (\int_{\mathbb{R}_+} \mathrm{e}^{-u^2/2}\,\mathrm{d}u)^2$ which is the result. □

3. Proof of Theorem 2.1

Clearly we can assume that $\int f = 1$. Let μ be the measure with density f. In the sequel we let $f_z(x) = f(z+x)$ for all x, z.

We prove the theorem by induction on the dimension. Let f be a non-negative Borel function on the line, let $r \in \mathbb{R}$ and $\lambda = \mu([r, \infty)) \in [0, 1]$. Let g satisfy $f(r+s)g(t) \leqslant \mathrm{e}^{-st}$, for all s, t. Apply Lemma 2.2 twice: first to $\phi_1(s) = f(r+s)$ and $\phi_2(t) = g(t)$ then to $\phi_1(s) = f(r-s)$ and $\phi_2(t) = g(-t)$. Then

$$\int_{\mathbb{R}_+} f_r \int_{\mathbb{R}_+} g \leqslant \frac{\pi}{2} \quad \text{and} \quad \int_{\mathbb{R}_-} f_r \int_{\mathbb{R}_-} g \leqslant \frac{\pi}{2}.$$

Therefore $\int_{\mathbb{R}_+} g \leqslant \frac{\pi}{2\lambda}$ and $\int_{\mathbb{R}_-} g \leqslant \frac{\pi}{2(1-\lambda)}$, which yields the result in dimension 1.

Assume the theorem to be true in dimension $n-1$. Let H be an affine hyperplane splitting $\mathbb{R}^n$ into two half-spaces H_+ and H_- and let $\lambda = \mu(H_+)$. Provided that $\lambda \neq 0, 1$ we can define b_+ and b_- to be the barycenters of $\mu_{|H_+}$ and $\mu_{|H_-}$, respectively. Since $\mu(H) = 0$, the point b_+ belongs to the interior of H_+, and similarly for b_-. Hence the line passing through b_+ and b_- intersects H at one point, which we call z. Let us prove that z satisfies (6), for all g. Clearly, replacing f by f_z and H by $H - z$, we can assume that $z = 0$. Let g satisfy

$$\forall x, y \in \mathbb{R}^n, \quad f(x)g(y) \leqslant \mathrm{e}^{-\langle x,y\rangle}. \tag{10}$$

Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $H = e_n^\perp$ and $\langle b_+, e_n\rangle > 0$. Let $v = b_+/\langle b_+, e_n\rangle$ and A be the linear operator on $\mathbb{R}^n$ that maps e_n to v and e_i to itself for $i = 1 \dots n-1$ and let $B = (A^{-1})^t$. Define

$$F_+ : y \in H \mapsto \int_{\mathbb{R}_+} f(y + sv)\,\mathrm{d}s \quad \text{and} \quad G_+ : y' \in H \mapsto \int_{\mathbb{R}_+} g(By' + te_n)\,\mathrm{d}t.$$

By Fubini, and since A has determinant 1, $\int_H F_+ = \int_{H_+} f \circ A = \mu(H_+) = \lambda$. Also, letting P be the projection with range H and kernel $\mathbb{R}v$, we have

$$\mathrm{bar}(F_+) = \frac{1}{\lambda}\int\limits_{H_+} P(Ax)f(Ax)\,\mathrm{d}x = \frac{1}{\lambda}P\Bigg(\int\limits_{H_+} xf(x)\,\mathrm{d}x\Bigg) = P(b_+),$$

and this is 0 by definition of P. Since $\langle Ax, Bx'\rangle = \langle x, x'\rangle$ for all $x, x' \in \mathbb{R}^n$, we have $\langle y+sv, By'+te_n\rangle = \langle y, y'\rangle + st$ for all $s, t \in \mathbb{R}$ and $y, y' \in H$. So (10) implies

$$f(y+sv)g(By'+te_n) \leqslant \mathrm{e}^{-st-\langle y,y'\rangle}.$$

Applying Lemma 2.2 to $\phi_1(s) = f(y+sv)$ and $\phi_2(t) = g(By'+te_n)$ we get $F_+(y)G_+(y') \leqslant \frac{\pi}{2}\mathrm{e}^{-\langle y,y'\rangle}$ for every $y, y' \in H$. Recall that $\mathrm{bar}(F_+) = 0$, then by the induction assumption (which implies Theorem 1.2 in dimension $n-1$)

$$\int\limits_H F_+ \int\limits_H G_+ \leqslant \frac{\pi}{2}(2\pi)^{n-1}, \tag{11}$$

hence $\int_{H_+} g(Bx)\,\mathrm{d}x \leqslant \frac{1}{4\lambda}(2\pi)^n$. In the same way $\int_{H_-} g(Bx)\,\mathrm{d}x \leqslant \frac{1}{4(1-\lambda)}(2\pi)^n$, adding these two inequalities, we obtain

$$\int\limits_{\mathbb{R}^n} g(Bx)\,\mathrm{d}x \leqslant \frac{1}{4\lambda(1-\lambda)}(2\pi)^n$$

which is the result since B has determinant 1.

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