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Harmonic Analysis

On Petermichl's dyadic shift and the Hilbert transform

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Abstract

Petermichl's representation for the Hilbert transform as an average of dyadic shifts has important applications. Here it is shown that the integrals involved in (a variant of) this representation converge both almost everywhere and strongly in $L^p(\mathbb{R})$, $p \in (1, \infty)$, which improves on the earlier result of weak convergence in $L^2(\mathbb{R})$. ***To cite this article: T. Hytönen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

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Résumé

La translation dyadique de Petermichl et la transformée d'Hilbert. La représentation, dû à Petermichl, pour la transformée d'Hilbert comme une moyenne des translations dyadiques a des applications importantes. Ici, on montre que les integrals dans (une forme de) cette représentation convergent à la fois presque partout et fortement dans $L^p(\mathbb{R})$, $p \in (1, \infty)$, ce qui améliore le résultat antérieur que affirme la convergence faible dans $L^2(\mathbb{R})$. ***Pour citer cet article : T. Hytönen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

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1. Introduction

Given a finite interval $I = [a, b) \subset \mathbb{R}$, denote its left and right halves by $I_- := [a, (a + b)/2)$ and $I_+ := [(a + b)/2, b)$. The associated standard Haar function is defined by $h_I := |I|^{-1/2}(1_{I_-} - 1_{I_+})$ and a modified Haar function by $H_I := 2^{-1/2}(h_{I_-} - h_{I_+})$, both normalized in $L^2(\mathbb{R})$.

Given a dyadic system of intervals $\mathcal{D}$ (what it means precisely, will be specified below), the associated dyadic shift, introduced by Petermichl [3], is defined by

$$\text{III} f := \sum_{I \in \mathcal{D}} H_I \langle h_I, f \rangle,$$

where $\langle g, f \rangle := \int_{-\infty}^{\infty} g(x) f(x)\, \mathrm{d}x$ and the letter III ('sha') stands for shift. The importance of this shift operator lies in the bridge which it provides between the probabilistic–combinatorial and the analytic realms. On the one hand, it is closely related to discrete martingale transforms and the simple structure of the operator makes it tractable to

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combinatorial considerations involving induction on the scales; on the other hand, the prototype singular integral operator, the Hilbert transform

$$Hf(x) := \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} H_{\varepsilon,R}f(x), \qquad H_{\varepsilon,R}f(x) := \frac{1}{\pi}\int\limits_{\varepsilon<|y|<R} \frac{f(x-y)}{y}\,\mathrm{d}y, \tag{1}$$

is, in a sense, an average of the dyadic shifts related to different dyadic systems, as shown in [3].

There are other representations of singular integrals in terms of transformations of the Haar basis, cf. [1], but an advantage of Petermichl's Ш is the strong localization property that the supports of Шh_I and h_I agree for all Haar functions h_I. This seems to be crucial when working in the weighted $L^p(w)$ spaces, with which many of the recent applications of the dyadic shift are involved [4,5].

With the recent success of the dyadic shifts, it seems interesting to look back at the precise connection of these operators and the Hilbert transform. What was actually shown in [3] is that $aH + M_b$, with some $a \in \mathbb{R}\setminus\{0\}$ and M_b the pointwise multiplication operator by a function $b \in L^\infty(\mathbb{R})$, lies in the weak operator closure in $\mathcal{L}(L^2(\mathbb{R}))$ of the convex hull of the dyadic shifts. However, it turns out that the Hilbert transform is an average of the dyadic shifts in a much stronger sense:

Theorem 1.1. *For $r \in [1,2)$ and $\beta \in \{0,1\}^{\mathbb{Z}}$, let $Ш^{\beta,r}$ be the dyadic shift associated to the dyadic system $r\mathcal{D}^\beta$, as defined in Section 2. Let μ stand for the canonical probability measure on $\{0,1\}^{\mathbb{Z}}$ which makes the coordinate functions β_j independent with $\mu(\beta_j = 0) = \mu(\beta_j = 1) = 1/2$. Then for all $p \in (1,\infty)$ and $f \in L^p(\mathbb{R})$,*

$$Hf(x) = -\frac{8}{\pi}\langle Ш\rangle f(x) := -\frac{8}{\pi}\int_1^2 \int\limits_{\{0,1\}^{\mathbb{Z}}} Ш^{\beta,r} f(x)\,\mathrm{d}\mu(\beta)\frac{\mathrm{d}r}{r},$$

where the integral converges both pointwise for a.e. $x \in \mathbb{R}$ and also in the sense of an $L^p(\mathbb{R})$-valued Bochner integral.

This is the present contribution, proven in the rest of the Note.

2. Dyadic systems and shifts

Recall that the standard dyadic system is

$$\mathcal{D}^0 := \bigcup_{j\in\mathbb{Z}} \mathcal{D}^0_j, \quad \mathcal{D}^0_j := \{2^j([0,1)+k):\ k\in\mathbb{Z}\}.$$

A general dyadic system may be defined as a collection $\mathcal{D} = \bigcup_{j\in\mathbb{Z}}\mathcal{D}_j$, where $\mathcal{D}_j = \mathcal{D}_j + x_j$ for some $x_j \in \mathbb{R}$ and the partition $\mathcal{D}_j$ refines $\mathcal{D}_{j+1}$ for each $j \in \mathbb{Z}$. Since only the value of $x_j \bmod 2^j$ is relevant for the definition of $\mathcal{D}_j$, one may choose $x_j \in [0, 2^j)$, and the refinement property requires that $x_{j+1} - x_j \in \{0, 2^j\}$. It follows by recursion that

$$x_j = \sum_{i<j} 2^i\beta_i, \quad \beta = (\beta_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}.$$

In this way, the set $\{0,1\}^{\mathbb{Z}}$ gives a natural parameterization of all dyadic systems as described. Denote

$$\mathcal{D}^\beta := \bigcup_{j\in\mathbb{Z}}\mathcal{D}^\beta_j, \quad \mathcal{D}^\beta_j := \mathcal{D}^0_j + \sum_{i<j}2^i\beta_i.$$

This parameterization is implicitly behind the notion of random dyadic systems as used by Nazarov, Treil and Volberg in the proof of their non-homogeneous Tb theorem [2, Sec. 9.1], but it was not similarly formulated there. When $\{0,1\}^{\mathbb{Z}}$ is equipped with the probability measure μ as defined in Theorem 1.1, one gets the same random distribution of dyadic systems as in their paper.

Note that the dyadic systems as defined here are more general than just translates of the standard system. The characteristic property of systems of the form $\mathcal{D}^0 + x$, $x \in \mathbb{R}$, is the existence of a special point, namely x, which does not belong to the interior of any dyadic interval $I \in \mathcal{D}^0 + x$, whereas for a general dyadic system such a point may or may not exist. One can check that $\mathscr{D}^\beta$ is of the form $\mathscr{D}^0 + x$ for some $x \in \mathbb{R}$ if and only if β_j is constant for large j; thus under the distribution μ, such systems have a vanishing probability.

Finally, one defines the scaled dyadic systems

$$r\mathscr{D}^\beta := \{rI = [ra, rb)\colon\ I = [a,b) \in \mathscr{D}^\beta\}, \quad r \in [1,2).$$

Observe the definition of rI above, which is not the central dilation commonly denoted like this in harmonic analysis. One could consider $r \in (0,\infty)$, but this would not add any new systems when both r and β are allowed to vary.

The dyadic shift associated to $r\mathscr{D}^\beta$ is

$$\text{III}^{\beta,r} f := \sum_{I \in r\mathscr{D}^\beta} H_I \langle h_I, f\rangle = \sum_{j\in\mathbb{Z}} \sum_{I \in r\mathscr{D}^\beta_j} H_I \langle h_I, f\rangle. \tag{2}$$

It follows easily from Burkholder's inequality for martingale transforms that partial sums of the j-series above give uniformly bounded operators on $L^p(\mathbb{R})$, $p \in (1,\infty)$; cf. [5, Theorem 3.1]. Then the convergence of the double series both in $L^p(\mathbb{R})$ and pointwise a.e. is deduced by standard considerations. Note that the partial sums of the j-series are differences of two conditional expectations of $\text{III}^{\beta,r} f$, and hence dominated by $M^{\beta,r}(\text{III}^{\beta,r} f)$, where $M^{\beta,r}$ is the dyadic maximal operator related to the scaled dyadic system $r\mathscr{D}^\beta$. From the boundedness of $M^{\beta,r}$ and $\text{III}^{\beta,r}$ on $L^p(\mathbb{R})$, with bounds independent of (β, r), one easily deduces by dominated convergence that the integral defining the average dyadic shift in Theorem 1.1 exists, pointwise a.e. and in $L^p(\mathbb{R})$, and satisfies

$$\langle \text{III}\rangle f := \lim_{\substack{n\to+\infty\\ m\to-\infty}} \sum_{j=m}^{n} \int_1^2 \int_{\{0,1\}^{\mathbb{Z}}} \sum_{I \in r\mathscr{D}^\beta_j} H_I \langle h_I, f\rangle \,\mathrm{d}\mu(\beta) \frac{\mathrm{d}r}{r}. \tag{3}$$

3. Evaluation of the integral

Observe that

$$r\mathscr{D}^\beta_j = r2^j \left\{ [0,1) + k + \sum_{i<j} 2^{i-j}\beta_i;\ k \in \mathbb{Z} \right\}.$$

When each of the numbers β_j is independently chosen from $\{0,1\}$, both values having equal probability, the binary expansion $u := \sum_{i<j} 2^{i-j}\beta_i$ is uniformly distributed over $[0,1)$. When, in addition, k runs through $\mathbb{Z}$, the sum $k+u$ runs through all of $\mathbb{R}$. With these observations, and introducing the new variable $t := 2^j r$, one transforms (3) into the form

$$\begin{aligned}\langle \text{III}\rangle f(x) &= \int_0^\infty \int_{-\infty}^\infty H_{t([0,1)+v)}(x) \langle h_{t([0,1)+v)}, f\rangle \,\mathrm{d}v \frac{\mathrm{d}t}{t},\\ &= \int_0^\infty \int_{-\infty}^\infty \frac{1}{t} \int_{-\infty}^\infty H_{[0,1)}\left(\frac{x}{t} - v\right) h_{[0,1)}\left(\frac{y}{t} - v\right) \mathrm{d}v\, f(y)\,\mathrm{d}y \frac{\mathrm{d}t}{t},\end{aligned}$$

where $\int_0^\infty$ is actually short-hand for the indefinite integral $\lim_{m,n\to\infty} \int_{2^{-m}}^{2^{+n}}$, and the limit exists in $L^p(\mathbb{R})$ as well as pointwise a.e.

The innermost integral above was already encountered and evaluated in [3]: as observed there, this is most easily done by recognizing it as the integral of the function $(\xi,\eta) \mapsto H_{[0,1)}(\xi) h_{[0,1)}(\eta)$ (which is piecewise constant on certain rectangles) along the straight line containing the point $(x/t, y/t)$ and having slope 1. The result depends only on $u := x/t - y/t$ and is the piecewise linear function $k(u)$ of this variable, which takes the values $0, -\frac{1}{4}, 0, \frac{3}{4}, 0, -\frac{3}{4}, 0, \frac{1}{4}, 0$ at the points $-1, -\frac{3}{4}, \ldots, \frac{3}{4}, 1$, interpolates linearly between them, and vanishes out off $(-1,1)$. So

$$\langle \text{III}\rangle f(x) = \int_0^\infty \int_{-\infty}^\infty \frac{1}{t} k\left(\frac{x}{t} - \frac{y}{t}\right) f(y)\,\mathrm{d}y \frac{\mathrm{d}t}{t} = \int_0^\infty k_t * f(x) \frac{\mathrm{d}t}{t},$$

where $k_t(x) := t^{-1}k(t^{-1}x)$ and $k * f$ are the usual L^1-dilation and convolution, respectively.

Denote $K(x) := \int_0^x k(u)\,\mathrm{d}u$ and $\phi(x) := x^{-1}K(x)1_{[-1,1]}(x)$. With this notation,

$$\int_\varepsilon^R k_t(x)\frac{\mathrm{d}t}{t} = \frac{1}{x}\big[K(x/\varepsilon) - K(x/R)\big] = \phi_\varepsilon(x) - \phi_R(x) - \frac{1}{8x}1_{\varepsilon<|x|<R},$$

and then

$$\int_\varepsilon^R k_t * f\frac{\mathrm{d}t}{t} = \phi_\varepsilon * f - \phi_R * f - \frac{\pi}{8}H_{\varepsilon,R}f,$$

where $H_{\varepsilon,R}$ is the truncated Hilbert transform as defined in (1).

As k is odd, its primitive K is even, and ϕ is again an odd function. It is also bounded and compactly supported. Thus it follows from standard mollification results that $\phi_t * f \to 0$, both in $L^p(\mathbb{R})$ and pointwise a.e., as either $t \to 0$ or $t \to \infty$. Hence

$$\langle \mathrm{III}\rangle f = -\frac{\pi}{8}\lim_{\substack{n\to+\infty\\ m\to-\infty}} H_{2^m,2^n}f =: -\frac{\pi}{8}Hf.$$

Note that the existence of the limit was a consequence of the proof, building at the bottom on martingale convergence; none of the well-known results concerning the $L^p(\mathbb{R})$ boundedness of the Hilbert transform nor the convergence of its truncated versions was presupposed, but all this follows as a byproduct of the established representation.

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