Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 346 (2008) 1119–1122
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Mathematical Problems in Mechanics

Control of irrigation channels with variable bathymetry and time dependent stabilization rate

Abdou Sene [a], Bocar Amadou Wane [b], Daniel Y. Le Roux [b]

[a] *UFR de sciences appliquées et technologie, Université Gaston-Berger, B.P. 234 Saint-Louis, Sénégal*
[b] *Département de mathématiques et statistique, Université Laval, G1K 7P4 Québec, Québec, Canada*

Received 18 October 2007; accepted after revision 4 September 2008

Available online 27 September 2008

Presented by Philippe G. Ciarlet

Abstract

This Note deals with the regulation of water flow in open-channels employing the shallow-water model with variable bathymetry. By using energy a priori estimation technics and the compactness theory, we build a stabilizing boundary control. The control law is based on an arbitrary choice of the time dependent stabilization rate r. Say, the energy decreases like the exponential of $-\int_0^t r(s)\,\mathrm{d}s$ when the time t tends to ∞. ***To cite this article: A. Sene et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Contrôle des canaux d'irrigation à bathymétrie variable et à vitesse de stabilisation dépendant du temps. Dans cette Note nous traitons un problème de contrôle des canaux d'irrigation à bathymétrie variable. En utilisant la technique d'estimation a priori de l'énergie et la théorie de la compacité, nous construisons un contrôle frontière. Ce dernier est basé sur le choix d'un taux de stabilisation r arbitraire et pouvant varier en fonction du temps. En effet, l'énergie décroît comme l'exponentielle de $-\int_0^t r(s)\,\mathrm{d}s$ quand le temps t tend vers ∞. ***Pour citer cet article : A. Sene et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

The shallow-water (SW) equations are the hyperbolic partial differential equations used in hydraulics to describe the behavior of the flow in open channels. The regulation of water level and water flow we are dealing with in this paper, can be done by using gate openings. This problem has been addressed for a long time in engineering in dynamics of canals and rivers. Interesting results have been published in the literature when the bathymetry has a constant or linear representation, and the stabilization rate is constant. Balogun et al. [1] and Malaterre [3] have developed LQ control on the basis of finite dimensional discrete approximations of the SW equations. Litrico and Georges [2], Xu and Sallet [5] have developed robust H_∞ control based on a simple model approximation using a linear diffusive wave equation. Further, Xu and Sallet [5] analyzed Boundary PI regulation on the basis of a linear PDE model around a

E-mail addresses: asene1001@yahoo.fr (A. Sene), bowane@mat.ulaval.ca (B.A. Wane), dleroux@mat.ulaval.ca (D.Y. Le Roux).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.09.009

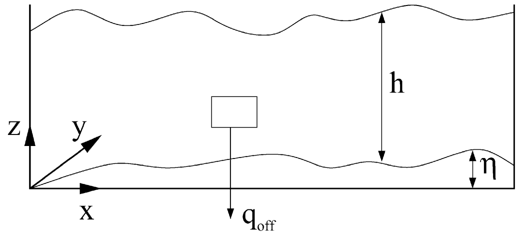

Fig. 1. A channel with discharge.

steady-state. In this paper, we establish a new method to build a controller in the case of an arbitrary bathymetry and time dependent stabilization rate. Our results are directly applicable to various types of control gates.

In Section 2 we perform the flow modeling in the case of a canal with arbitrary bed shape and the model equations are obtained. In Section 3, we establish the main results of this paper. By using a priori energy estimation technics, a controller is proposed, depending on an arbitrary nonnegative function $r(t)$ which represents the stabilization rate. The energy decreases like the exponential of $-\int_0^t r(s)\,\mathrm{d}s$ when the time t tends to ∞. This stabilization rate is only required to verify $\int_0^\infty r(t)\,\mathrm{d}t = +\infty$. Some concluding remarks complete the study.

2. Governing equations in open channels

The one dimensional SW model used in the present paper is derived from the incompressible Navier–Stokes (NS) model. Let h be the height of the fluid column, q the flow, and η the bathymetry function, as shown in Fig. 1. The model equations are

$$L_y\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=-\frac{q_{\mathrm{off}}}{l_{\mathrm{off}}},\qquad L_y\frac{\partial q}{\partial t}+gL_y^2h\frac{\partial(h+\eta)}{\partial x}+\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right)+\frac{(\alpha^2-1)q^2}{2h^2}\frac{\partial\eta}{\partial x}+\frac{q_{\mathrm{off}}}{l_{\mathrm{off}}}\frac{q}{h}=0, \tag{1}$$

where q_{off} is the offtake flow along the canal, l_{off} is the length of the irrigation hole, L_x and L_y are respectively the length and the width of the canal, and the correction coefficient α is given by $\alpha=\frac{\text{bottom velocity}}{\text{mean velocity}}$. The boundary conditions are $q(t,0)=q_0(t)$, $q(t,L_x)=q_{L_x}(t)$, and the initial ones read $h(0,x)=h^0(x)$, $q(0,x)=q^0(x)$. Note that for linear representations of the bathymetry, (1) is similar to that obtained in [4] where S stands for L_yh, and $\Phi(S,Q,q)$ corresponds to $\frac{(\alpha^2-1)q^2}{2h^2}\frac{\partial\eta}{\partial x}+\frac{q_{\mathrm{off}}}{l_{\mathrm{off}}}\frac{q}{h}$. Note that $\frac{\alpha^2-1}{2}\frac{\partial\eta}{\partial x}$ characterizes the nature of the bottom represented by $-g\bar{H}\frac{\bar{R}^{3/4}}{K^2}$ in [4].

2.1. Steady-state

The stabilization process consists in finding controllers, say inflows and outflows, such that the water level and velocity always approach the steady state denoted by $(\bar{h},\bar{q})$. The spatial derivatives of $\bar{h}$ and $\bar{q}$ are obtained from (1) as

$$\text{(a)}\ \frac{\partial\bar{q}}{\partial x}=-\frac{q_{\mathrm{off}}}{l_{\mathrm{off}}},\qquad \text{(b)}\ \frac{\partial\bar{h}}{\partial x}=-\frac{\left(-\frac{(\alpha^2-1)\bar{q}^2}{2\bar{h}^2}-gL_y^2\bar{h}\right)\frac{\partial\eta}{\partial x}+\frac{\bar{q}}{\bar{h}}\frac{q_{\mathrm{off}}}{l_{\mathrm{off}}}}{\frac{\bar{q}^2}{\bar{h}^2}-gL_y^2\bar{h}}. \tag{2}$$

Integrating (2a) leads to $\bar{q}(x)=\bar{q}_0+\bar{Q}_{\mathrm{off}}(x)$, with $\bar{q}_0=\bar{q}(x=0)$ and $\bar{Q}_{\mathrm{off}}(x)=-\int_0^x\frac{q_{\mathrm{off}}(s)}{l_{\mathrm{off}}}\,\mathrm{d}s$. Let $\alpha>0$ and assume that the data q_{off}, l_{off}, $\bar{q}_0$ and the bathymetry function $\eta(\cdot)$ are such that $\bar{h}>\alpha$, e.g. this assumption is met if the source flow $\bar{q}_0$ is greater than the offtake flow $|\bar{Q}_{\mathrm{off}}(\cdot)|$, γ_1 (defined in (5)) is positive, and $\frac{\partial\eta}{\partial x}<0$. Since the stabilization is performed in order to keep the solution (h,q) of (1) close enough to the steady state, then h is expected to be positive.

3. The controller building process

Let $(\bar{h}, \bar{q})$ be the steady state of system (1). We introduce the residual state $(\tilde{h}, \tilde{q})$ as the difference between the present state (h, q) and the steady state $(\bar{h}, \bar{q})$, and hence $\tilde{h}(x,t) = h(x,t) - \bar{h}(x)$, $\quad \tilde{q}(x,t) = q(x,t) - \bar{q}(x)$. Then, using the assumptions $\tilde{h} \ll \bar{h}$ and $\tilde{q} \ll \bar{q}$ to linearize (1), we deduce that $(\tilde{h}, \tilde{q})$ is the solution of

$$L_y \frac{\partial \tilde{h}(x,t)}{\partial t} + \frac{\partial \tilde{q}(x,t)}{\partial x} = 0, \tag{3}$$

$$L_y \frac{\partial \tilde{q}(x,t)}{\partial t} + \beta_1(x) \frac{\partial \tilde{q}(x,t)}{\partial x} + \beta_0(x)\tilde{q}(x,t) + \gamma_1(x) \frac{\partial \tilde{h}(x,t)}{\partial x} + \gamma_0(x)\tilde{h}(x,t) = 0, \tag{4}$$

with the controllers $\tilde{q}(t,0) = 0$ and $\tilde{q}(t, L_x) = \tilde{q}_{L_x}(t)$, and the initial conditions $\tilde{h}(0,x) = \tilde{h}^0(x)$ and $\tilde{q}(0,x) = \tilde{q}^0(x)$, where the functions β_0, $\gamma_0 \beta_1$, γ_1 are defined as

$$\beta_0 = \frac{1}{\bar{h}} \left(2\frac{\partial \bar{q}}{\partial x} - 2\frac{\bar{q}}{\bar{h}} \frac{\partial \bar{h}}{\partial x} + \frac{\alpha^2 - 1}{2} \frac{\bar{q}}{\bar{h}} \frac{\partial \eta}{\partial x} + \frac{q_{\mathrm{off}}}{l_{\mathrm{off}}} \right), \quad \beta_1 = 2\frac{\bar{q}}{\bar{h}}, \quad \gamma_0 = g L_y^2 \frac{\partial(\bar{h} + \eta)}{\partial x}, \quad \gamma_1 = g L_y^2 \bar{h} - \frac{\bar{q}^2}{\bar{h}^2}. \tag{5}$$

In the sequel we assume $\gamma_1 > 0$, and $\beta_1(L_x) \geqslant 0$, i.e. there is outflow on $\{x = L_x\}$. The condition on γ_1 is equivalent to the subcritical or fluvial flow, i.e. $|\frac{\bar{q}}{L_y \bar{h}}| \leqslant \sqrt{g\bar{h}}$. That is, the flow velocity is less than the wave velocity.

In [5] Xu proved that (3), (4) is equivalent to an hyperbolic symmetric system. He showed that the system is exponentially stable by using the semi-group method. Here, we generalize the results obtained in [5] by considering a variable bathymetry and a time dependent stabilization rate.

3.1. A priori energy estimation and controller building

The following system is the weak formulation of (3)–(4): $\forall (\Psi, \Phi) \in H^1(]0, L_x[)$,

$$\int_0^{L_x} L_y \gamma_1 \Psi \frac{\partial \tilde{h}}{\partial t} \,\mathrm{d}x - \int_0^{L_x} \tilde{q} \frac{\partial (\gamma_1 \Psi)}{\partial x} \,\mathrm{d}x + \gamma_1(L_x)\Psi(L_x)\tilde{q}_{L_x}(t) = 0, \tag{6}$$

$$\int_0^{L_x} L_y \Phi \frac{\partial \tilde{q}}{\partial t} \,\mathrm{d}x + \int_0^{L_x} \beta_1 \Phi \frac{\partial \tilde{q}}{\partial x} \,\mathrm{d}x + \int_0^{L_x} \beta_0 \tilde{q} \Phi \,\mathrm{d}x + \int_0^{L_x} \gamma_1 \Phi \frac{\partial \tilde{h}}{\partial x} \,\mathrm{d}x + \int_0^{L_x} \gamma_0 \tilde{h} \Phi \,\mathrm{d}x = 0, \tag{7}$$

with $\tilde{h}(x,0) = \tilde{h}^0(x)$ and $\tilde{q}(x,0) = \tilde{q}^0(x)$. Let us estimate the variation of the energy

$$E(t) = \int_0^{L_x} L_y \big(\gamma_1 \tilde{h}^2(t) + \tilde{q}^2(t) \big) \,\mathrm{d}x$$

in order to define the controller q_{L_x} on $\{x = L_x\}$. For this purpose, let $(\Psi, \Phi) = (\tilde{h}, \tilde{q})$ in (6)–(7). Hence,

$$\frac{1}{2} \frac{\partial}{\partial t} E(t) = a(\tilde{q}_{L_x})^2 + b\,(q_{L_x}) + c, \quad \text{where} \tag{8}$$

$$a = -\frac{1}{2}\beta_1(L_x), \quad b = -\gamma_1(L_x)\tilde{h}(t, L_x), \quad c = \int_0^{L_x} \left(\frac{\partial \gamma_1}{\partial x} - \gamma_0 \right) (\tilde{q}\tilde{h}) \,\mathrm{d}x + \frac{1}{2} \int_0^{L_x} \left(\frac{\partial \beta_1}{\partial x} - 2\beta_0 \right) \tilde{q}^2 \,\mathrm{d}x. \tag{9}$$

The RHS of (8) is a second order polynomial Q with respect to $\tilde{q}_{L_x}$. Let $r(t)$ be any positive real function, and $\tilde{q}_{L_x}$ be the real part of

$$\max\big(\mathrm{sign}(b), 0\big) \frac{-b + \sqrt{b^2 - 4a(c + rE)}}{2a} + \max\big(-\mathrm{sign}(b), 0\big) \frac{-b - \sqrt{b^2 - 4a(c + rE)}}{2a}. \tag{10}$$

Note that c is bounded by the energy E, and then the controller $\tilde{q}_{L_x}$ defined in (10) necessarily vanishes when E tends to 0. If r is continuous and very large at 0, $\tilde{q}_{L_x}$ may be also. However, in any case $\tilde{q}_{L_x}$ exponentially tends to zero.

Since a is negative, i.e. $\beta_1(L_x)$ is positive, we have $Q(\tilde{q}_{L_x}) \leqslant -r(t)E(t)$. Hence, the energy decreases exponentially with a rate depending on the function $r(\cdot)$ which can be chosen arbitrarily. Indeed, Eq. (8) gives $\frac{1}{2}\frac{\partial}{\partial t}E(t) = Q(\tilde{q}_{L_x}) \leqslant -r(t)E(t)$, i.e.

$$\int_0^{L_x} L_y\big(\gamma_1 \tilde{h}^2(t) + \tilde{q}^2(t)\big)\,\mathrm{d}x \leqslant \exp\left(-2\int_0^t r(s)\,\mathrm{d}s\right)\int_0^{L_x} L_y\big(\gamma_1(\tilde{h}^0)^2(x) + (\tilde{q}^0)^2(x)\big)\,\mathrm{d}x.$$

Theorem 3.1. *With the feedback law* (10)*, the decay rate of the order of the function* $r(\cdot)$ *given a priori arbitrarily is guaranteed.*

Proof. The result is achieved by combining the a priori energy estimation and the Galerkin method. □

4. Conclusions

In this Note, a general sufficient stabilization condition for flow and water levels in open channels has been described and analyzed around the steady state. A control law design based on this stabilization condition is applied to open channels. The main theoretical result of the paper is the generalization based on time dependent stabilization rate with variable bathymetry.

Note that this work is done on a linearized version of (1). However, we are also investigating the nonlinear case. As far as this latter is concerned, the energy a priori estimation is achieved, but the main issue one faces is the convergence of finite dimensional problems in the Galerkin method for existence of solution. In future research we will also address the 2D SW control using the same approach.

Acknowledgements

This work is supported by grants from the Fonds National Suisse pour la Recherche (FNSR), the Natural Sciences and Engineering Research Council (NSERC), Fonds Québécois de la Recherche sur la Nature et les Technologies (FQRNT) and the Dept. of Math. of Georgia Institute of Technology.

References

[1] O. Balogun, M. Hubbard, J. De Vries, Automatic control of canal flow using linear quadratic regulator theory, Journal of Hydraulic Engineering 119 (1988) 615–630.

[2] X. Litrico, Robust flow control of single input multiple output regulated rivers, Journal of Irrigation and Drainage Engineering 127 (2001) 281–286.

[3] P. Malaterre, Pilot: linear quadratic optimal controller for irrigation canals, Journal of Irrigation and Drainage Engineering 124 (1998) 187–194.

[4] M. Ndiaye, G. Bastin, Commande frontière adaptative d'un bief de canal avec prélèvements inconnus, RS-JESA 38 (2004) 347–371.

[5] C.Z. Xu, G. Sallet, Proportional and integral regulation of irrigation canal systems governed by the Saint-Venant equation, in: 14th IFAC World Congress, Beijing, China, 1999.