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Mathematical Analysis/Partial Differential Equations

Lifting default for $\mathbb{S}^1$-valued maps

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Abstract

Let $\varphi \in C^\infty([0,1]^N, \mathbb{R})$. When $0 < s < 1$, $p \geqslant 1$ and $1 \leqslant sp < N$, the $W^{s,p}$-semi-norm $|\varphi|_{W^{s,p}}$ of φ is not controlled by $|g|_{W^{s,p}}$, where $g := e^{\imath\varphi}$ [J. Bourgain, H. Brezis, P. Mironescu, Lifting in Sobolev spaces, J. Anal. Math. 80 (2000) 37–86]. [This question is related to existence, for $\mathbb{S}^1$-valued maps g, of a lifting φ as smooth as allowed by g.] In [J. Bourgain, H. Brezis, P. Mironescu, Lifting, degree, and distributional Jacobian revisited, Commun. Pure Appl. Math. 58 (2005) 529–551], the authors suggested that $|g|_{W^{s,p}}$ does control a weaker quantity, namely $|\varphi|_{W^{s,p}+W^{1,sp}}$. Existence of such control is due to J. Bourgain and H. Brezis [J. Bourgain, H. Brezis, On the equation div $Y = f$ and application to control of phases, J. Amer. Math. Soc. 16 (2003) 393–426] when $1 < p \leqslant 2$, $s = 1/p$ and to H.-M. Nguyen [H.-M. Nguyen, Inequalities related to liftings and applications, C. R. Acad. Sci. Paris, Ser. I 346 (17–18) (2008) 957–962] when $N = 1$, $p > 1$ and $sp \geqslant 1$ or when $N \geqslant 2$, $p > 1$ and $sp > 1$. In this Note, we establish existence of control for all $s < 1$, $p \geqslant 1$ and N. ***To cite this article: P. Mironescu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Défaut de relèvement pour les applications à valeurs dans le cercle unité. Soit $\varphi \in C^\infty([0,1]^N, \mathbb{R})$. Si $0 < s < 1$, $p \geqslant 1$ et $1 \leqslant sp < N$, alors la semi-norme $|\varphi|_{W^{s,p}}$ n'est pas contrôlée par $|g|_{W^{s,p}}$, où $g := e^{\imath\varphi}$ [J. Bourgain, H. Brezis, P. Mironescu, Lifting in Sobolev spaces, J. Anal. Math. 80 (2000) 37–86]. [Cette question est liée à l'existence, pour des g à valeurs dans $\mathbb{S}^1$, de relèvements φ aussi réguliers que g le permet.] Dans [J. Bourgain, H. Brezis, P. Mironescu, Lifting, degree, and distributional Jacobian revisited, Commun. Pure Appl. Math. 58 (2005) 529–551], il est conjecturé que $|g|_{W^{s,p}}$ contrôle une quantité plus faible que $|\varphi|_{W^{s,p}}$, plus spécifiquement $|\varphi|_{W^{s,p}+W^{1,sp}}$. L'existence d'un tel contrôle est due à J. Bourgain and H. Brezis [J. Bourgain, H. Brezis, On the equation div $Y = f$ and application to control of phases, J. Amer. Math. Soc. 16 (2003) 393–426] pour $1 < p \leqslant 2$ et $s = 1/p$ et à H.-M. Nguyen [H.-M. Nguyen, Inequalities related to liftings and applications, C. R. Acad. Sci. Paris, Ser. I 346 (17–18) (2008) 957–962] pour $N = 1$, $p > 1$ et $sp \geqslant 1$ ou pour $N \geqslant 2$, $p > 1$ et $sp > 1$. Dans cette Note, nous montrons l'existence d'un contrôle pour tout $s < 1$, $p \geqslant 1$ et N. ***Pour citer cet article : P. Mironescu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

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Soient $C = [0,1]^N$, $0 < s < \infty$ et $1 \leqslant p < \infty$. $|\ |_{W^{s,p}}$ désigne une semi-norme standard sur l'espace de Sobolev $W^{s,p}(C)$; par exemple, pour $0 < s < 1$ nous considérons la semi-norme de Gagliardo,

$$|u|_{W^{s,p}} = \left(\iint_{C^2} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p}.$$

Si $\varphi \in C^\infty(C, \mathbb{R})$ et $g := e^{\imath\varphi}$, alors $|\nabla\varphi| = |\nabla g|$. En particulier, $|g|_{W^{1,p}} = |\varphi|_{W^{1,p}}$. Plus généralement, $|g|_{W^{s,p}}$ contrôle $|\varphi|_{W^{s,p}}$ si $s \geqslant 1$, au sens où il existe une inégalité de la forme $|\varphi|_{W^{s,p}} \leqslant F(|g|_{W^{s,p}})$ avec F croissante [3]. Ceci n'est plus forcément vrai si $0 < s < 1$. Voici un exemple inspiré de [3] : si $N \geqslant 2$ et si $0 < s < 1$ et p sont tels que $1 < sp < N$, alors il existe une fonction $\psi \in W^{1,sp} \setminus W^{s,p}$. Si on considère $\varphi_\varepsilon := \psi * \rho_\varepsilon$, avec ρ noyau régularisant, alors $|\varphi_\varepsilon|_{W^{s,p}} \to \infty$ (car $\psi \notin W^{s,p}$), alors que $g_\varepsilon := e^{\imath\varphi_\varepsilon}$ reste bornée dans $W^{1,sp} \cap L^\infty$ (car $\psi \in W^{1,sp}$) et donc dans $W^{s,p}$, grâce à l'inclusion de Gagliardo–Nirenberg $W^{1,sp} \cap L^\infty \subset W^{s,p}$. Nonobstant la non inclusion $W^{1,1} \cap L^\infty \not\subset W^{s,p}$ si $sp = 1$, on peut adapter cet exemple au cas où $sp = 1$ et $N \geqslant 1$. Plus généralement, si $sp = 1$ ou $1 < sp < N$, alors on peut obtenir l'absence du contrôle à partir d'une fonction convenable $\psi \in W^{s,p} + W^{1,sp}$.

Dans le cas particulier $s = 1/2$ et $p = 2$, J. Bourgain et H. Brezis [2] ont montré que le contre-exemple ci-dessus est essentiellement le seul. Leur résultat est que toute fonction φ se décompose comme $\varphi = \varphi_1 + \varphi_2$, où $|\varphi_1|_{H^{1/2}}$ et $|\varphi_2|_{W^{1,1}}$ sont contrôlées par $|g|_{H^{1/2}}$. La preuve s'étend aux espaces $W^{1/p,p}$ avec $1 < p \leqslant 2$ et donne la décomposition (1). Ce résultat a motivé le problème suivant [4,8] avec $0 < s < 1$ et $1 \leqslant p < \infty$:

$(D_{s,p})$ Tout $\varphi \in C^\infty(C, \mathbb{R})$ s'écrit $\varphi = \varphi_1 + \varphi_2$, où $|\varphi_1|_{W^{s,p}} \leqslant C|e^{\imath\varphi}|_{W^{s,p}}$ et $\|D\varphi_2\|_{L^{sp}} \leqslant C|e^{\imath\varphi}|_{W^{s,p}}^{1/s}$.

Récemment, H.-M. Nguyen [10] a résolu ce problème lorsque $p > 1$, $s = 1/p$ et $N = 1$; son argument s'applique aussi au cas $N \geqslant 2$, $p > 1$ et $sp > 1$.

Le but de cette Note est d'annoncer le suivant :

Théorème 1. *La décomposition* $(D_{s,p})$ *est valide pour tout* $0 < s < 1$, $p \geqslant 1$ *et* N.

Notons que la décomposition existe même si $sp < 1$.

Idée de la preuve. On étend g à une application $h : \mathbb{R}^N \to \mathbb{R}^2$, de sorte que h soit Lipschitz, constante en dehors d'un compact, $|h| \leqslant 3$ et $|h|_{W^{s,p}(\mathbb{R}^N)} \leqslant C|g|_{W^{s,p}(C)}$. On étend ensuite h à $\mathbb{R}^N \times \mathbb{R}_+^*$ par la formule $w(x,\varepsilon) = \Pi(h * \rho_\varepsilon(x))$. Ici, $\Pi \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ est telle que $\Pi(z) = z/|z|$ si $|z| \geqslant 1/2$, tandis que $\rho \in C_0^\infty$ satisfait $\rho \geqslant 0$, $\int \rho = 1$, $\operatorname{supp} \rho \subset B(0,2) \setminus B(0,1)$. Alors la conclusion du théorème est vérifiée par φ_j données par

$$\varphi_1(x) := -\int_0^\infty w(x,\varepsilon) \wedge \frac{\partial w}{\partial \varepsilon}(x,\varepsilon) \, \mathrm{d}\varepsilon, \qquad \varphi_2 := \varphi - \varphi_1.$$

L'inégalité $|\varphi_1|_{W^{s,p}} \leqslant C|h|_{W^{s,p}}$ découle d'estimations standard[1] pour les régularisées $h * \rho_\varepsilon$ d'une fonction $h \in W^{s,p}$. Elle implique $|\varphi_1|_{W^{s,p}} \leqslant C|g|_{W^{s,p}}$.

Pour estimer $D\varphi_2$, le point de départ est l'identité[2]

$$D\varphi_2(x) = -2 \int_0^\infty \frac{\partial}{\partial \varepsilon} w(x,\varepsilon) \wedge D_x w(x,\varepsilon) \, \mathrm{d}\varepsilon.$$

En adaptant une idée de [4], cette identité permet d'obtenir l'estimation $\|D\varphi_2\|_{L^{sp}} \leqslant C|e^{\imath\varphi}|_{W^{s,p}}^{1/s}$. □

[1] L'une des estimations utilisées dans la preuve est $\int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p-sp-1} |D[h * \rho_\varepsilon](x)|^p \, \mathrm{d}\varepsilon \, \mathrm{d}x \leqslant C|h|_{W^{s,p}}^p$, bien connue par les spécialistes [11], II.12.

[2] Évidente, du moins formellement.

Le lecteur trouvera les preuves détaillées dans [9] ; entre autres, on y explique pourquoi notre décomposition est une sorte d'analogue continu de la décomposition trouvée dans [2].

1. Introduction

Let $C = [0,1]^N$, $0 < s < \infty$ and $1 \leqslant p < \infty$. We will denote by $|\ |_{W^{s,p}}$ a standard semi-norm on the Sobolev space $W^{s,p}(C)$; e.g., when $0 < s < 1$, we take the Gagliardo semi-norm

$$|u|_{W^{s,p}} = \left(\iint\limits_{C^2} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}.$$

Let $\varphi \in C^\infty(C, \mathbb{R})$ and set $g = e^{\imath\varphi}$. Since $|\nabla\varphi| = |\nabla g|$, we have $|g|_{W^{1,p}} = |\varphi|_{W^{1,p}}$. More generally, $|g|_{W^{s,p}}$ controls $|\varphi|_{W^{s,p}}$ when $s \geqslant 1$, i.e., there is some nondecreasing F such that $|\varphi|_{W^{s,p}} \leqslant F(|g|_{W^{s,p}})$ [3]. This need not hold when $0 < s < 1$. Here is an example, essentially taken from [3]: let $N \geqslant 2$ and let $0 < s < 1$, $1 \leqslant p < \infty$ be such that $1 < sp < N$. By a Sobolev "nonembedding", there is some $\psi \in W^{1,sp} \setminus W^{s,p}$. Let $\varphi_\varepsilon := \psi * \rho_\varepsilon$ and set $g_\varepsilon = e^{\imath\varphi_\varepsilon}$, where ρ is a mollifier. Then $|\varphi_\varepsilon|_{W^{s,p}} \to \infty$ (since $\psi \notin W^{s,p}$). On the other hand, $g_\varepsilon := e^{\imath\varphi_\varepsilon}$ is bounded in $W^{1,sp} \cap L^\infty$ (since $\psi \in W^{1,sp}$) and thus in $W^{s,p}$, by the Gagliardo–Nirenberg embedding $W^{1,sp} \cap L^\infty \subset W^{s,p}$. Despite the nonembedding $W^{1,1} \cap L^\infty \not\subset W^{s,p}$ when $sp = 1$, one may easily adapt this example to the case $sp = 1$ and $N \geqslant 1$. More generally, when $sp = 1$ or $1 < sp < N$, one may prove lack of control with the help of an appropriate $\psi \in W^{s,p} + W^{1,sp}$.

In the special case $s = 1/2$ and $p = 2$, J. Bourgain and H. Brezis [2] proved that the above counter-example is essentially the only one. Their results asserts that each φ splits as $\varphi = \varphi_1 + \varphi_2$, where $|\varphi_1|_{H^{1/2}}$ and $|\varphi_2|_{W^{1,1}}$ are controlled by $|g|_{H^{1/2}}$. Their argument adapts steadily to the spaces $W^{1/p,p}$ where $1 < p \leqslant 2$ and yields

$$\begin{aligned}&\text{If } 1 < p \leqslant 2, \text{ then } \varphi = \varphi_1 + \varphi_2\\ &\quad\text{where } \varphi_j \in C^\infty(C),\ |\varphi_1|_{W^{1/p,p}} \leqslant C|g|_{W^{1/p,p}},\ |\varphi_2|_{W^{1,1}} \leqslant C|g|^p_{W^{1/p,p}}.\end{aligned} \tag{1}$$

This motivated the following open problem [4,8]:

$(D_{s,p})$ Each $\varphi \in C^\infty(C, \mathbb{R})$ splits as $\varphi = \varphi_1 + \varphi_2$, where $|\varphi_1|_{W^{s,p}} \leqslant C|e^{\imath\varphi}|_{W^{s,p}}$ and $\|D\varphi_2\|_{L^{sp}} \leqslant C|e^{\imath\varphi}|^{1/s}_{W^{s,p}}$.

[Here, $0 < s < 1$ and $1 \leqslant p < \infty$.] Very recently, H.-M. Nguyen [10] answered positively this problem when $p > 1$, $s = 1/p$ and $N = 1$; his argument adapts to the case where $N \geqslant 2$, $p > 1$ and $sp > 1$.

The main purpose of this Note is to announce the following:

Theorem 1. *The decomposition* $(D_{s,p})$ *holds for each* $0 < s < 1$, $p \geqslant 1$ *and* N.

Unlike the proofs in [2,10], our method applies to the case $sp < 1$.

2. Heuristics of the proof of Theorem 1

In order to explain the main idea, we consider, for simplicity, maps defined on $\mathbb{R}^N$ which are constant at infinity (rather than maps defined on $[0,1]^N$). Assume first that φ has small amplitude oscillations, say $|\varphi| \ll 1$. Then $|g - 1| \ll 1$ and $|\varphi|_{W^{s,p}} \sim |g|_{W^{s,p}}$. Thus, in this case, a convenient decomposition is $\varphi_1 = \varphi$ and $\varphi_2 = 0$. We next proceed as follows: we derive a formula for φ. This formula gives φ only when φ has small amplitude oscillations, but we may give it a meaning for each φ. In general (i.e., when φ may oscillate), we take this formula as the definition of φ_1 and simply let φ_2 be the phase excess, i.e., we set $\varphi_2 = \varphi - \varphi_1$.

In order to obtain a tractable formula for φ, we rely on the following remark: if $g = e^{\imath\varphi}$, then there is no formula giving φ in terms of g, but there is one for $D\varphi$, since $D\varphi = g \wedge Dg$. We consider a smooth extension $w : \mathbb{R}^N \times [0, +\infty[\to \mathbb{S}^1$ of g such that

$$\lim_{\varepsilon \to \infty} w(., \varepsilon) = \text{const.} \tag{2}$$

A natural choice is $w(x,\varepsilon) = \Pi(g * \rho_\varepsilon(x))$, where $\Pi(z) = z/|z|$ and ρ is a mollifier. This yields a smooth map whenever g (and thus $g * \rho_\varepsilon$) is close to 1. We may write $w = e^{\imath\psi}$, where $\psi(x,0) = \varphi(x)$. Assuming that convergence in (2) is sufficiently fast, we than have $\psi(x,\infty) = C$ and thus

$$\varphi(x) = -\psi(x,\varepsilon)\big|_{\varepsilon=0}^{\varepsilon=\infty} + C = C - \int_0^\infty w(x,\varepsilon) \wedge \frac{\partial w}{\partial \varepsilon}(x,\varepsilon)\,\mathrm{d}\varepsilon. \tag{3}$$

As explained in the next section, (3) gives the right definition of φ_1, provided we pick an appropriate ρ and we change slightly the definition of Π.

3. Sketch of the proof of Theorem 1

Let $\varphi \in C^\infty(C,\mathbb{R})$ and set $g = e^{\imath\varphi}$. We first extend g to a Lipschitz map $h:\mathbb{R}^N \to \mathbb{R}^2$ such that h is constant outside $[-1,2]^N$, $|h| \leqslant 3$ and $|h|_{W^{s,p}(\mathbb{R}^N)} \leqslant C|g|_{W^{s,p}(C)}$. We next extend h to $\mathbb{R}^N \times \mathbb{R}_+^*$ through the formula $w(x,\varepsilon) = \Pi(h * \rho_\varepsilon(x))$. Here, $\Pi \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ is such that $\Pi(z) = z/|z|$ if $|z| \geqslant 1/2$, while $\rho \in C_0^\infty$ satisfies $\rho \geqslant 0$, $\int \rho = 1$, $\operatorname{supp} \rho \subset B(0,2) \setminus B(0,1)$. Then the conclusion of the theorem holds with φ_j given by

$$\varphi_1(x) := -\int_0^\infty w(x,\varepsilon) \wedge \frac{\partial w}{\partial \varepsilon}(x,\varepsilon)\,\mathrm{d}\varepsilon, \qquad \varphi_2 := \varphi - \varphi_1. \tag{4}$$

The inequality

$$|\varphi_1|_{W^{s,p}} \leqslant C|h|_{W^{s,p}} \tag{5}$$

follows from standard estimates[1] for regularizations $h * \rho_\varepsilon$ of maps $h \in W^{s,p}$. As a consequence of (5), we find that $|\varphi_1|_{W^{s,p}} \leqslant C|g|_{W^{s,p}}$.

[Estimate (5) is a cousin of the estimate

$$\left| x \mapsto \int_0^\infty u * \rho_\varepsilon(x) \frac{\partial}{\partial \varepsilon}\big[v * \rho_\varepsilon(x)\big]\,\mathrm{d}\varepsilon \right|_{W^{s,p}} \leqslant C\|u\|_{L^\infty}|v|_{W^{s,p}}, \tag{6}$$

valid for any reasonable ρ and presumably well known to experts. For special ρ's, the "discrete" and much more popular analog of (6) is the "paraproduct inequality" [7]

$$\left| \sum_{j \leqslant k} u_j v_k \right|_{W^{s,p}} \leqslant C\|u\|_{L^\infty}|v|_{W^{s,p}}, \tag{7}$$

where $u = \sum u_j$, $v = \sum_j v_j$ are the Littlewood–Paley decompositions of u and v. Both (6) and (7) are refinements of the standard inequality $\|uv\|_{W^{s,p}} \leqslant C(\|u\|_{W^{s,p}}\|v\|_{L^\infty} + \|v\|_{W^{s,p}}\|u\|_{L^\infty})$.]

The starting point for estimating $D\varphi_2$ is the identity

$$D\varphi_2(x) = -2\int_0^\infty \frac{\partial}{\partial \varepsilon} w(x,\varepsilon) \wedge D_x w(x,\varepsilon)\,\mathrm{d}\varepsilon. \tag{8}$$

At least formally, this identity follows from

$$D\varphi_2(x) = D\varphi(x) + \int_0^\infty D_x w(x,\varepsilon) \wedge \frac{\partial}{\partial \varepsilon} w(x,\varepsilon)\,\mathrm{d}\varepsilon + \int_0^\infty w(x,\varepsilon) \wedge \frac{\partial}{\partial \varepsilon} D_x w(x,\varepsilon)\,\mathrm{d}\varepsilon$$

[1] One of the key estimates used in the proof is $\int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p-sp-1} |D[h * \rho_\varepsilon](x)|^p \,\mathrm{d}\varepsilon\,\mathrm{d}x \leqslant C|h|^p_{W^{s,p}}$, well known to experts [11], II.12.

$$= w(x,\varepsilon) \wedge D_x w(x,\varepsilon)_{|\varepsilon=0} + \int_0^\infty D_x w(x,\varepsilon) \wedge \frac{\partial}{\partial\varepsilon} w(x,\varepsilon)\,\mathrm{d}\varepsilon + w(x,\varepsilon) \wedge D_x w(x,\varepsilon)\Big|_{\varepsilon=0}^{\varepsilon=\infty}$$

$$- \int_0^\infty \frac{\partial}{\partial\varepsilon} w(x,\varepsilon) \wedge D_x w(x,\varepsilon)\,\mathrm{d}\varepsilon = -2\int_0^\infty \frac{\partial}{\partial\varepsilon} w(x,\varepsilon) \wedge D_x w(x,\varepsilon)\,\mathrm{d}\varepsilon.$$

Adapting an idea from [4], this identity implies the estimate $\|D\varphi_2\|_{L^{sp}} \leqslant C|e^{\imath\varphi}|_{W^{s,p}}^{1/s}$. □

The interested reader will find the detailed proof in [9].

We end this section by comparing our decomposition to the Bourgain–Brezis one. Assuming, for the sake of the simplicity, that φ and g are defined on the whole $\mathbb{R}^N$, the decomposition in [2] is (with $g = \sum g_j$ the Littlewood–Paley decomposition of g)

$$\varphi_1 := \sum_{j \leqslant k} g_j \wedge g_k, \qquad \varphi_2 := \varphi - \varphi_1. \tag{9}$$

In the same way (6) is related to (7), one may interpret the formula defining φ_1 in (9) as a discrete analog of

$$x \mapsto -\int_0^\infty g * \rho_\varepsilon(x) \wedge \frac{\partial}{\partial\varepsilon}\big[g * \rho_\varepsilon(x)\big]\,\mathrm{d}\varepsilon.$$

Thus our decomposition is a continuous version of the one in [2], with the additional sophistication that the regularizations of g are "almost projected" onto $\mathbb{S}^1$ (via Π).

4. Some applications

As a first application, we may achieve the description of $X^{s,p} = \overline{C^\infty(C;\mathbb{S}^1)}^{W^{s,p}}$, partly obtained in [3] and [5].

Theorem 2. *Let* $0 < s < \infty$, $1 \leqslant p < \infty$. *Then*

(a) *([3]) When* $sp < 1$ *or* $sp \geqslant N$, $X^{s,p} = W^{s,p}(C;\mathbb{S}^1)$;
(b) *([5]) When* $s \geqslant 1$ *and* $sp \geqslant 2$, $X^{s,p} = W^{s,p}(C;\mathbb{S}^1)$;
(c) *([5]) When* $s \geqslant 1$ *and* $1 \leqslant sp < 2$, $X^{s,p} = \{e^{\imath\varphi}; \varphi \in W^{s,p} \cap W^{1,sp}(C,\mathbb{R})\}$;
(d) *When* $0 < s < 1$ *and* $1 < sp < N$, $X^{s,p} = \{e^{\imath\varphi}; \varphi \in (W^{s,p} + W^{1,sp})(C,\mathbb{R})\}$;
(e) *When* $0 < s < 1$, $N \geqslant 2$ *and* $sp = 1$, $X^{s,p} = \{e^{\imath\varphi}; \varphi \in (W^{s,p} + W^{1,1})(C,\mathbb{R})\} \cap W^{s,p}(C;\mathbb{S}^1)$.

The lack of symmetry between statements (d) and (e) is explained by the fact that, when $sp > 1$ and $s < 1$, we have $\varphi \in W^{1,sp} \Longrightarrow e^{\imath\varphi} \in W^{s,p}$; this implication fails when $sp = 1$ and $s < 1$. For the proof of (d) and (e), we send the reader to [9].

In a forthcoming joint paper with H. Brezis and H.-M. Nguyen [6], we investigate several consequences of Theorem 1. We end this Note by mentioning one of them.

Theorem 3. *(See [6].) Assume that* $N \geqslant 3$, $0 < s < 1$, $2 \leqslant sp < N$, $k \in \{2, 3, \ldots\}$. *Then, for each* $u \in W^{s,p}(C;\mathbb{S}^1)$, *there is some* $v \in W^{s,p}(C;\mathbb{S}^1)$ *such that* $u = v^k$.

This answers a question of F. Bethuel and D. Chiron [1]. [For the other values of s, p and N, the surjectivity of the map $v \mapsto v^k$ is clarified in [1]. The case considered in Theorem 3 is the only one left open in [1].]

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