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Optimal Control

Finite-time partial stabilizability of chained systems

Chaker Jammazi

Laboratoire d'ingénierie mathématique, École polytechnique de Tunisie, B.P. 743, La Marsa 2078, Tunis, Tunisia

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Abstract

The Note deals with partial stabilization in finite-time of a class of nonlinear chained systems. It is well known that the chain of integrators of length n is not asymptotic stabilizable by continuous stationary feedback laws. This follows from the Brockett necessary condition for stabilizability. To overcome this limitation, we construct feedback laws that stabilize in finite-time the $(n-1)$ first components of this chain of integrators while the last component converges. This special stabilization is obtained by continuous feedback laws and smooth outside the origin. ***To cite this article: C. Jammazi, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Stabilisation partielle en temps fini des systèmes chaînés. On considère des systèmes chaînés qui peuvent modéliser différents systèmes d'origine mécanique ou biologique. On sait depuis Brockett que cette classe de systèmes, qui est contrôlable, n'est pas stabilisable par des feedbacks statiques et continus. Pour contourner le problème, nous proposons l'approche de la stabilisation partielle en temps fini. Nous construisons dans ce travail des feedbacks permettant d'annuler en temps fini les $(n-1)$ premières composantes tout en assurant la convergence de la dernière composante. Les feedbacks obtenus sont continus et réguliers en dehors de zéro. ***Pour citer cet article : C. Jammazi, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Soit le système $\dot{x} = f(x,u)$ où $x \in \mathbb{R}^n$ est l'état du système, $u \in \mathbb{R}^m$ le contrôle et f est de classe C^∞ de $\mathbb{R}^n \times \mathbb{R}^m$ à valeurs dans $\mathbb{R}^n$ et vérifie $f(0,0) = 0$. On sait depuis les travaux de Brockett [2], que la contrôlabilité n'implique pas l'existence d'un retour d'état $u(x)$, nul en 0 et tel que le point d'équilibre $x = 0$ soit asymptotiquement stable pour le système bouclé $\dot{x} = f(x, u(x))$. Pour contourner cette obstruction, deux voies possibles ont été adoptées. La première direction repose sur l'utilisation des retours d'état instationnaires. Dans la seconde voie plusieurs auteurs ont développé l'approche par retours d'état discontinus, et parfois discontinus instationnaires. Pour plus de détails sur ces deux voies, nous renvoyons le lecteur au livre de Coron [7].

Cependant, l'utilisation des retours d'état instationnaires, qui fait introduire le temps dans la loi de bouclage, peut produire des oscillations "indésirables" du système au voisinage de son point d'équilibre. Pour remédier à cet

E-mail address: Chaker.Jammazi@ept.rnu.tn.

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inconvénient, ainsi que pour contourner l'obstruction de Brockett, nous proposons la stratégie de la stabilisation asymptotique partielle. Cette approche consiste à utiliser des retours d'état stationnaires et continus pour maintenir le maximum de composantes de l'état asymptotiquement stabilisables, et de faire converger les autres composantes vers des constantes qui dépendent des données initiales.

Pour énoncer nos résultats, nous introduisons la définition suivante :

Définition 0.1. Soit le système de contrôle

$$\dot{x}_1 = f_1(x, u), \quad \dot{x}_2 = f_2(x, u), \tag{1}$$

où $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ est l'état du système, $u \in \mathbb{R}^m$ le contrôle. On suppose que $f := (f_1, f_2) \in C^\infty$, que, pour tout $x_2 \in \mathbb{R}^{n-p}$, $f_1((0, x_2), 0) = 0$ et que $f_2((0, x_2), 0) = 0$.

Le système (1) est dit p-partiellement stabilisable en temps fini, s'il existe une commande $x \mapsto u(x)$ continue, tel que, pour tout $x_2 \in \mathbb{R}^{n-p}$, $u(0, x_2) = 0$, et, pour le système bouclé $\dot{x} = f(x, u(x))$,

- $0 \in \mathbb{R}^n$ est stable au sens de Lyapunov.
- Il existe un temps fini $T > 0$ et un nombre $r > 0$ tels que si $|x(0)| < r$, alors $x_1(t) = 0$ $\forall t \geqslant T$ (et donc $x_2(t)$ est constant pour tout $t \geqslant T$).

Si, de plus, pour tout $x(0)$ il existe un temps T tel que $x_1(t) = 0$ $\forall t \geqslant T$, on dit que le système (1) est globalement p-partiellement stabilisable en temps fini.

Un des principaux objectifs de cette Note est de prouver le résultat suivant :

Théorème 0.2. *Le système chaîné*

$$\dot{x}_1 = x_2 u_1, \quad \dot{x}_2 = x_3 u_1, \quad \ldots, \quad \dot{x}_{n-3} = x_{n-2} u_1, \quad \dot{x}_{n-2} = x_{n-1} u_1, \quad \dot{x}_{n-1} = u_2, \quad \dot{x}_n = u_1, \tag{2}$$

où l'état est $(x_1, \ldots, x_n) \in \mathbb{R}^n$ et le contrôle est $(u_1, u_2) \in \mathbb{R}^2$, est globalement $(n-1)$-partiellement stabilisable en temps fini.

Un exemple classique et important de système sans dérive, est le système de contrôle qui décrit le mouvement de l'unicycle [13,15]. Il est modélisé par le système d'équations suivant

$$\dot{x}_1 = u_1 \cos\theta, \quad \dot{x}_2 = u_1 \sin\theta, \quad \dot{\theta} = u_2, \tag{3}$$

où $x = (x_1, x_2, \theta) \in \mathbb{R}^3$ est l'état du système et $u = (u_1, u_2) \in \mathbb{R}^2$ est le contrôle.

Le problème de stabilisation de l'unicycle, en tant qu'exemple de système sans dérive de dimension trois, a fait l'objet de nombreuses recherches ; voir, en particulier, [9,8,13,14] où l'on trouvera des commandes instationnaires régulieres stabilisant le système (3). (Voir aussi [5,6] pour des systèmes généraux.)

Pour démontrer le Théorème 0.2, on décompose le contrôle u_2 comme suit : $u_2 = u_1 u$. On pose $x = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, donc x vérifie l'équation $\dot{x} = u_1 g(x, u)$ avec $g(x, u) = (x_1, x_2, \ldots, x_{n-1}, u)$. En se basant sur les travaux de Huang et al. [11], on sait que le système $\dot{x} = g(x, u)$ est globalement stabilisable en temps fini. Cette stabilisabilité (en temps fini) est obtenue moyennant un feedback continu u (non lipschtzien), et grâce à une fonction de Lyapunov V construite de manière récursive. La fonction V est de classe C^1, vérifie la relation $\dot{V} + cV^\alpha \leqslant 0$, avec $\alpha = \frac{2(n-1)}{2(n-1)+1}$. Il suffit alors de choisir $u_1 = V^\beta$ où $\alpha + \beta < 1$ et de conclure enfin avec le Théorème de Bhat et al. [1].

1. Introduction

A classical benchmark example of systems without drift is the unicycle system [13,15]. This system is described by Eqs. (3): the state is given by the orientation θ, together with the coordinates (x_1, x_2) of the midpoint between the back wheels. $u = (u_1, u_2) \in \mathbb{R}^2$ is the control. We consider the control u_1 as a “drive command” and u_2 as a steering control. The stabilization problem of the unicycle is well studied and focused a number of publications. The first one is established by Samson [13] where the conception of time-varying feedback laws stabilizing the unicycle are demonstrated. See also [8,9].

It is well known that the system (3) is globally controllable but cannot be stabilized by means of stationary feedback laws. This from the classical Brockett condition. To get around the problem of impossibility to stabilize many controllable systems by continuous feedback laws, two main strategies have been proposed:

- Asymptotic stabilization by means of discontinuous feedback laws-see e.g. the pioneer work by Sussmann [17] as well as [10,4], and the references therein,
- Asymptotic stabilization by means of continuous time-varying feedback laws – see e.g. the pioneer works by Sontag et al. [16] and by Samson [13] as well as [6–8,5,12,9] and the references therein.

The main goal of this Note is to propose a new strategy based on the construction of continuous feedback laws that stabilize the first components in finite-time while the other components converge. This concept is the "*finite-time partial stabilizability*".

We adopt the following definition:

Definition 1.1. Let $X : \mathbb{R}^p \times \mathbb{R}^{n-p} \cong \mathbb{R}^n \to \mathbb{R}^n$, $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p} \mapsto X(x) \in \mathbb{R}^n$, be defined and continuous on a neighborhood of $(0, 0) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$. We assume that

$$X(0, x_2) = 0.$$

One says that $(0, 0) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ is p-partially stable in finite-time for $\dot{x} = X(x)$ if

- $(0, 0)$ is Lyapunov stable for $\dot{x} = X(x)$,
- there exists $r > 0$ and $T > 0$, such that, if $\dot{x} = X(x)$ and $|x(0)| < r$, then $x_1(t) = 0$ for every $t \geqslant T$ (and therefore $t \mapsto x_2(t)$ is constant for $t \geqslant T$).

The control system (1) is p-partially stabilizable in finite-time if there exist a continuous feedback $x \mapsto u(x)$ such that, for every $x_2 \in \mathbb{R}^{n-p}$, $u(0, x_2) = 0$, and such that $(0, 0) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ is p-partially stable in finite-time for the closed loop system $\dot{x} = f(x, u(x))$.

A key ingredient for the partial stabilization in finite-time of chained systems is the following proposition, where $\mathcal{K}$ denotes the set of the continuous functions $\alpha : [0, +\infty) \in [0, +\infty)$ which are strictly increasing and vanish at 0.

Proposition 1. *Let us consider the control system* (1). *Let us assume that there exist* $\varepsilon > 0$, *a function* $V : \mathbb{R}^p \times \mathbb{R}^{n-p} \to \mathbb{R}$ *of class* C^1, *and a function* $u \in C^0(\mathbb{R}^m, \mathbb{R}^n)$ *such that* $u(0, x_2) = 0$, *satisfying*:

(i) $\exists \alpha_1, \alpha_2 \in \mathcal{K}$ *such that, for every* $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ *such that* $|x| < \varepsilon$,

$$\alpha_1\big(|x_1|\big) \leqslant V(x_1, x_2) \leqslant \alpha_2\big(|x_1|\big), \tag{4}$$

(ii) *there exists* $c > 0$ *and* $a \in (0, 1)$ *such that, for every* $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ *such that* $|x| < \varepsilon$,

$$\dot{V}(x_1, x_2) := \nabla V(x) \cdot f\big(x, u(x)\big) \leqslant -c\big(\alpha_2\big(|x_1|\big)\big)^a. \tag{5}$$

Then $(0, 0) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ *is p-partially stable in finite-time for the closed loop system* $\dot{x} = f(x, u(x))$.

Proof of Proposition 1. Conditions (i) and (ii) imply that $0 \in \mathbb{R}^n$ is Lyapunov stable for the closed loop system $\dot{x} = f(x, u(x))$ (see [18, Theorem 0.4.1, pp. 26] for instance).

By combining the assertions (i) and (ii), we obtain the existence of a constants $a \in (0, 1)$ and $c > 0$ such that

$$\dot{V} \leqslant -cV^a. \tag{6}$$

Using the Lyapunov stability of $0 \in \mathbb{R}^n$ for $\dot{x} = f(x, u(x))$ and (6), one gets the existence of $r > 0$ and $T > 0$, such that, if $\dot{x} = X(x)$, $t \geqslant T$ and $|x(0)| < r$,

$$V\big(x_1(t), x_2(t)\big) = 0,$$

which, together with (4), implies that $x_1(t) = 0$. This concludes the proof of Proposition 1. □

1.1. Stabilizing feedback laws for the unicycle

In this section we deal with the unicycle system (3), a benchmark example of controllable chained system that cannot be stabilized by continuous stationary state feedback law. Our goal is to construct continuous feedback laws u_1 and u_2 such that $u_1(0,0,\theta) = u_2(0,0,\theta) = 0$ and, for the closed-loop system, $(0,0) \in \mathbb{R}^2 \times \mathbb{R}$ is 2-partially stable in finite-time.

As usual, let us introduce the new coordinates [15]

$$z_1 = x_1 \sin\theta - x_2 \cos\theta, \quad z_2 = x_1 \cos\theta + x_2 \sin\theta, \quad z_3 = \theta, \quad v_1 = u_2, \quad v_2 = u_1 - z_1 u_2. \tag{7}$$

Simple calculation yields

$$\dot z_1 = v_1 z_2, \quad \dot z_2 = v_2, \quad \dot z_3 = v_1. \tag{8}$$

Note that $(x_1, x_2) = 0$ is equivalent to $(z_1, z_2) = 0$. Based on the pioneer work [1] by Bhat et al., we show how to stabilize in finite-time the system (8) with respect to (z_1, z_2), while the state z_3 converges in finite-time. These feedback controllers are given in the following proposition

Proposition 2. *Let be $\alpha \in (0,1)$, and let the feedback*

$$u = -\operatorname{sign}(z_2)|z_2|^\alpha - \operatorname{sign}\bigl(\phi(z_1,z_2)\bigr)\bigl|\phi(z_1,z_2)\bigr|^{\frac{\alpha}{2-\alpha}}, \quad \phi(z_1,z_2) = z_1 + \frac{1}{2-\alpha}\operatorname{sign}(z_2)|z_2|^{2-\alpha}. \tag{9}$$

Let the candidate Lyapunov function

$$V(z_1,z_2) = \frac{2-\alpha}{3-\alpha}\bigl|\phi(z_1,z_2)\bigr|^{\frac{3-\alpha}{2-\alpha}} + s z_2 \phi(z_1,z_2) + \frac{r}{3-\alpha}|z_2|^{3-\alpha}, \quad r > 0,\ s > 0. \tag{10}$$

Then $(0,0) \in \mathbb{R}^2 \times \mathbb{R}$ is 2-partially stable in finite-time for the closed loop system (8) with v_1 and v_2 defined by

$$v_1(z) := \bigl(V(z_1,z_2)\bigr)^\beta, \quad \beta > 0 \text{ such that } \gamma := \beta + \frac{2}{3-\alpha} < 1,\ v_2 := v_1 u.$$

Proof of Proposition 2. The time-derivative of V along the system (8) is given by

$$\dot V = \frac{\partial V}{\partial z_1} z_2 v_1 + \frac{\partial V}{\partial z_2} u v_1 = v_1 \dot V|_{(12)}, \tag{11}$$

where $\dot V|_{(12)}$ denotes the time derivative of V for the following system:

$$\dot z_1 = z_2, \qquad \dot z_2 = u. \tag{12}$$

Thanks to [1, Proposition 1] , $\dot V|_{(12)}$ satisfies for all $(z_1,z_2) \in \mathbb{R}^2$ the inequality

$$\dot V|_{(12)} \leqslant -c V^{\frac{2}{3-\alpha}}, \quad c > 0.$$

Then, if we choose the feedback $v_1 \geqslant 0$, the inequality (11) becomes

$$\dot V \leqslant -c v_1 V^{\frac{2}{3-\alpha}}. \tag{13}$$

Then one can take

$$v_1 = V^\beta, \quad \text{where } \beta > 0 \text{ such that } \gamma := \beta + \frac{2}{3-\alpha} \in (0,1). \tag{14}$$

With this v_1, we get

$$\dot V \leqslant -c V^\gamma \leqslant 0,$$

which, together with Proposition 1, concludes the proof of Proposition 2. □

Numerical simulation. The simulation (see Fig. 1) shows the stabilization in finite-time the components z_1, z_2 and the convergence in finite-time of z_3.

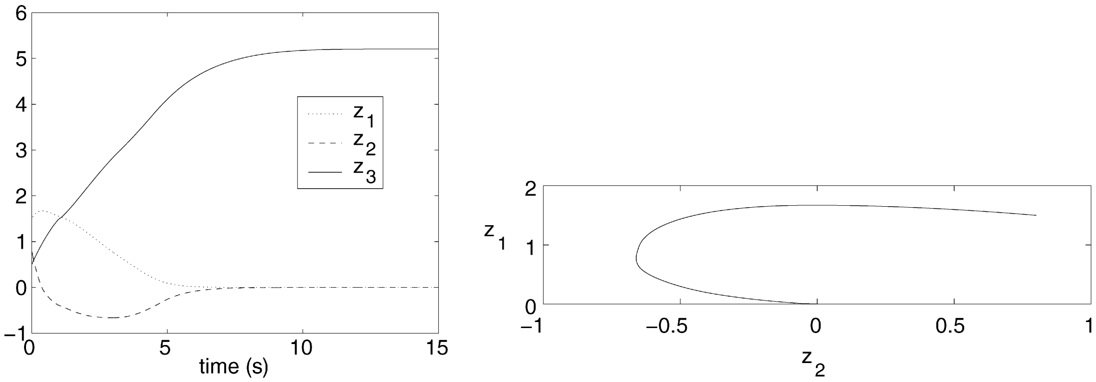

Fig. 1. (Left) trajectories of the state z_1, z_2 and z_3; (right) motion in the plane (z_2, z_1).

Remark 3. The partial asymptotic stability in finite-time of the system (8) is practical. Indeed, the component z_3 of the state is the angle that gives the orientation of the vehicle. We have shown with the action of our feedback that we can put in finite-time the vehicle in the equilibrium position without taking in consideration this angle (this later converges to some position depending on the initial conditions). Clearly, for many cases, this type of partial stabilization is sufficient.

2. Finite-time partial stabilizability of chained systems

In this section we extended the previous results to all chained systems:

Theorem 2.1. *The control system* (2) *is* $(n-1)$*-partially stabilizable in finite-time.*

Proof of Theorem 2.1. We decompose the input u_2 with the following form $u_2 = uu_1$, where u is a suitable feedback law which stabilize in finite-time the chain of integrators

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_4, \quad \ldots, \quad \dot{x}_{n-3} = x_{n-2}, \quad \dot{x}_{n-2} = x_{n-1}, \quad \dot{x}_{n-1} = u. \tag{15}$$

All the assumptions of [11, Theorem 1] hold. Therefore, for the control system (15), there exist a continuous, feedback law $x \mapsto u(x)$, a Lyapunov function V constructed with an explicit recursive design, and two real numbers $\alpha \in (0,1)$ and $c > 0$ such that

$$\dot{V}|_{(15)} + cV^{\alpha} \leqslant 0, \quad \alpha = \frac{2(n-1)}{2(n-1)+1},$$

where $\dot{V}|_{(15)}$ denotes the time derivative of V along the system (15). The time derivative of the function V for system (2) is given by

$$\dot{V} = u_1 \dot{V}|_{(15)}. \tag{16}$$

If $u_1 \geqslant 0$, then (16) becomes

$$\dot{V} = u_1 \dot{V}|_{(15)} \leqslant -cu_1 V^{\alpha}. \tag{17}$$

Now, we choose the feedback u_1 with the following form

$$u_1 = V^{\beta}, \quad \text{where } \beta > 0 \text{ satisfies } \gamma := \alpha + \beta < 1.$$

We have $\dot{V} \leqslant -cV^{\gamma}$. As for unicycle case, we conclude by using Proposition 1. □

The following theorem extends Theorem 2.1 to all multi-chained system:

Theorem 2.2. *Let m be a positive integer, let $n_1, \ldots, n_m$ be m non negative integers, and let $n = 1 + m + \sum_{j=1}^{m} n_j$. The following m-chain single-generator chained form* [3]

$$\begin{cases} \dot z_{1,0} = v_1, & \dot z_{2,0} = v_2, & & \dot z_{m,0} = v_m, \\ \dot z_{1,1} = v_0 z_{1,0}, & \dot z_{2,1} = v_0 z_{2,0}, & & \dot z_{m,1} = v_0 z_{m,0}, \\ \vdots & \vdots & \ldots & \vdots \\ \dot z_{1,n_1} = v_0 z_{1,n_1-1}, & \dot z_{2,n_2} = v_0 z_{2,n_2-1}, & & \dot z_{m,n_m} = v_0 z_{m,n_m-1}, & \dot z_n = v_0, \end{cases}$$

is $(n-1)$*-partially stabilizable in finite-time* (*the last component of the state being* z_n).

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