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Probability Theory

A necessary and sufficient condition for invertibility of adapted perturbations of identity on Wiener space

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Abstract

Let (W, H, μ) be the classical Wiener space, assume that $U = I_W + u$ is an adapted perturbation of identity satisfying the Girsanov identity. Then, U is invertible if and only if the kinetic energy of u is equal to the relative entropy of the measure induced with the action of U on the Wiener measure μ, in other words U is invertible if and only if

$$\frac{1}{2}\int_W |u|_H^2 \,\mathrm{d}\mu = \int_W \frac{\mathrm{d}U\mu}{\mathrm{d}\mu} \log \frac{\mathrm{d}U\mu}{\mathrm{d}\mu}\,\mathrm{d}\mu.$$

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Résumé

Une condition nécessaire et suffisante pour l'inversibilité de perturbations d'identité adaptées sur l'espace de Wiener. Soit (W, H, μ) l'espace de Wiener classique, et soit $U = I_W + u$ une perturbation d'identité adaptée satisfaisant á l'identité de Girsanov. Alors U est inversible si et seulement si l'énergie cinétique de u est égale à l'entropie relative de la mesure induite par l'action de U sur la mesure de Wiener μ ; en d'autres termes U est inversible si et seulement si

$$\frac{1}{2}\int_W |u|_H^2 \,\mathrm{d}\mu = \int_W \frac{\mathrm{d}U\mu}{\mathrm{d}\mu} \log \frac{\mathrm{d}U\mu}{\mathrm{d}\mu}\,\mathrm{d}\mu.$$

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Version française abrégée

Soit (W, H, μ) l'espace de Wiener classique : $W = C([0, 1], \mathbb{R}^d)$, $d \geqslant 1$ μ est la mesure gaussienne standard et H est l'espace de Cameron–Martin dont le produit scalaire et la norme sont notés respectivement par $(h, k)_H = \int_0^1 \dot{h}_s \cdot \dot{k}_s \,\mathrm{d}s$ et par $|\cdot|_H$. Nous noterons par $(\mathcal{F}_t, t \in [0, 1])$ la filtration canonique du mouvement brownien canonique,

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eventuellement complétée. Soit maintenant $u \in L_a^0(\mu, H)$, où cette dernière représente les classes d'équivalences de variables aléatoires à valeurs dans H telles que leurs dérivées temporelles sont adaptées à la filtration brownienne. On notera par $\rho(u)$ l'exponentielle de Girsanov définie par

$$\rho(u) = \exp\left(\int_0^1 \dot{u}_s \, \mathrm{d}W_s - \frac{1}{2}\int_0^1 |\dot{u}_s|^2 \, \mathrm{d}s\right).$$

Nous avons le résultat suivant :

Théorème 1. *Supposons que $E[\rho(-u)] = 1$ et notons par U l'application $I_W + u$. Alors les propriétés suivantes sont équivalentes* :

1. *L'application U est p.s. inversible, son inverse peut s'écrire comme $V = I_W + v$ avec $v \in L_a^0(\mu, H)$,*
2. *L'équation differentielle stochastique suivante* :
$$\mathrm{d}V_t = -\dot{u}_t \circ V \, \mathrm{d}t + \mathrm{d}W_t,$$
$$V_0 = 0$$
possède une solution (forte) unique,
3. *On a l'identité*
$$\frac{1}{2}\int_W |u|_H^2 \, \mathrm{d}\mu = \int_W \frac{\mathrm{d}U\mu}{\mathrm{d}\mu} \log \frac{\mathrm{d}U\mu}{\mathrm{d}\mu} \, \mathrm{d}\mu.$$

L'équivalence entre (1) et (2) a déjà démontrée dans [6] et le reste de cet article sera consacré à la preuve de l'équivalence entre (1) et (3).

1. Main results

Let (W, H, μ) the classical Wiener space on $\mathbb{R}^d$, we begin with the following proposition whose proof follows from the Girsanov theorem:

Proposition 1. *Assume that $L = \rho(-v)$, where $v \in L_a^0(\mu, H)$, i.e., $\dot{v}$ is adapted and $\int_0^1 |\dot{v}_s|^2 \, \mathrm{d}s < \infty$ a.s. Then there exists $U = I_W + u$, with $u : W \to H$ adapted such that $U\mu = L\mu$ and $E[\rho(-u)] = 1$ if and only if the following condition is satisfied:*

$$1 = L_t \circ U E\left[\rho(-u^t)|\mathcal{U}_t\right] \tag{1}$$
$$= L_t \circ U E\left[\rho(-u)|\mathcal{U}_t\right] \tag{2}$$

almost surely for any $t \in [0, 1]$, where u^t is defined as $u^t(\tau) = \int_0^{t\wedge\tau} \dot{u}_s \, \mathrm{d}s$ and $\mathcal{U}_t$ is the sigma algebra generated by $(w(\tau) + u(\tau), \tau \leqslant t)$.

Let us calculate $E[\rho(-u^t)|\mathcal{U}_t] = E[\rho(-u)|\mathcal{U}_t]$ in terms of the innovation process associated to U. Recall that the term innovation, which originates from the filtering theory is defined as (cf. [2] and [4])

$$Z_t = U_t - \int_0^t E[\dot{u}_s|\mathcal{U}_s] \, \mathrm{d}s$$

and it is a μ-Brownian motion with respect to the filtration $(\mathcal{U}_t, t \in [0, 1])$. A proof similar to the one in [2] shows that any martingale with respect to the filtration of U can be represented as a stochastic integral with respect to Z. Hence, by the positivity assumption, $E[\rho(-u)|\mathcal{U}_t]$ can be written as an exponential martingale

$$E\left[\rho(-u)|\mathcal{U}_t\right] = \exp\left(-\int_0^t (\dot{\xi}_s, \mathrm{d}Z_s) - \frac{1}{2}\int_0^t |\dot{\xi}_s|^2 \, \mathrm{d}s\right).$$

A double utilization of the Girsanov theorem gives the following explicit result:

Proposition 2. *We have*

$$E\big[\rho(-\delta u)|\mathcal{U}\big] = \exp\left(-\int_0^1 \big(E[\dot{u}_s|\mathcal{U}_s], \mathrm{d}Z_s\big) - \frac{1}{2}\int_0^1 \big|E[\dot{u}_s|\mathcal{U}_s]\big|^2 \,\mathrm{d}s\right), \tag{3}$$

and

$$E\big[\rho(-\delta u)|\mathcal{U}_t\big] = \exp\left(-\int_0^t \big(E[\dot{u}_s|\mathcal{U}_s], \mathrm{d}Z_s\big) - \frac{1}{2}\int_0^t \big|E[\dot{u}_s|\mathcal{U}_s]\big|^2 \,\mathrm{d}s\right), \tag{4}$$

almost surely.

Combining Propositions 1 and 2, we obtain

Theorem 2. *A necessary and sufficient condition for the relation* (1) *is that*

$$E[\dot{u}_t|\mathcal{U}_t] = -\dot{v}_t \circ U$$

$\mathrm{d}t \times \mathrm{d}\mu$*-almost surely.*

Now we state and prove the main theorem (cf. also [1] for related problems):

Theorem 3. *Assume that $u \in L^2(\mu, H) \cap L^0_a(\mu, H)$ with $E[\rho(-u)] = 1$. Define L as*

$$L = \frac{\mathrm{d}U\mu}{\mathrm{d}\mu} = \rho(-v)$$

where $v \in L^0_a(\mu, H)$ is given by the Itô representation theorem. The map $U = I_W + u$ is then almost surely invertible with its inverse $V = I_W + v$ if and only if

$$2E[L \log L] = E\big[|u|^2_H\big].$$

In other words, U is invertible if and only if

$$H(U\mu|\mu) = \frac{1}{2}\|u\|^2_{L^2(\mu,H)},$$

where $H(U\mu|\mu)$ denotes the entropy of $U\mu$ with respect to μ.

Proof. Since U represents $Ld\mu$, we have, from Theorem 2, $E[\dot{u}_s|\mathcal{U}_s] + \dot{v}_s \circ U = 0$ $\mathrm{d}s \times \mathrm{d}\mu$-almost surely. Hence, from the Jensen inequality $E[|v \circ U|^2_H] \leqslant E[|u|^2_H]$. Moreover, the Girsanov theorem gives

$$2E[L \log L] = E\big[|v|^2_H L\big] = E\big[|v \circ U|^2_H\big] = E\left[\int_0^1 \big|E[\dot{u}_s|\mathcal{U}_s]\big|^2 \,\mathrm{d}s\right].$$

Hence the hypothesis implies that

$$E\big[|u|^2_H\big] = E\left[\int_0^1 \big|E[\dot{u}_s|\mathcal{U}_s]\big|^2 \,\mathrm{d}s\right].$$

From this we deduce that $\dot{u}_s = E[\dot{u}_s|\mathcal{U}_s]\,\mathrm{d}s \times \mathrm{d}\mu$-almost surely. Finally we get $\dot{u}_s + \dot{v}_s \circ U = 0\,\mathrm{d}s \times \mathrm{d}\mu$, which is a necessary and sufficient condition for the invertibility of U (cf. [4–6]). The necessity is obvious. □

Corollary 1. *With the notations of theorem, U is not invertible if and only if we have*

$$\frac{1}{2}E\big[|u|_H^2\big] > H(U\mu|\mu).$$

Remark. This result gives an enlightenment about the celebrated counter example of Tsirelson, cf. [3].

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