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Differential Geometry

The length of a shortest geodesic loop

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Abstract

We give a lower bound for the length of a non-trivial geodesic loop on a simply-connected and compact manifold of even dimension with a non-reversible Finsler metric of positive flag curvature. Harris and Paternain use this estimate in their recent paper to give a geometric characterization of dynamically convex Finsler metrics on the 2-sphere. ***To cite this article: H.-B. Rademacher, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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Résumé

La longueur d'un lacet géodésique. On donne une borne inférieure pour la longueur d'un lacet géodésique non-triviale sur une variété compacte et simplement connexe munie d'une métrique de Finsler non-reversible de courbure positive. Harris et Paternain utilisent cette éstimée dans leur récent article afin de donner und charactérisation géométrique des métriques de Finsler à convexité dynamique sur la sphère de dimension 2. ***Pour citer cet article : H.-B. Rademacher, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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On a compact and simply-connected Riemannian manifold with positive sectional curvature $0 < K \leqslant 1$ the length of a non-constant geodesic loop is bounded from below by 2π. This result is due to Klingenberg [3] and is of importance in proofs of the classical sphere theorem.

For a compact manifold M with non-reversible Finsler metric F the author introduced in [5] the *reversibility* $\lambda := \max\{F(-X);\, F(X) = 1\} \geqslant 1$. In this short Note we show how one can use the results and methods from [5] to obtain the following estimate for the length of a geodesic loop depending on the flag curvature and the reversibility:

Proposition 1. *Let M be a compact and simply-connected differentiable manifold of even dimension $n \geqslant 2$ equipped with a non-reversible Finsler metric F and flag curvature K satisfying $0 < K \leqslant 1$. Then the length l of a shortest non-constant geodesic loop is bounded from below: $l \geqslant \pi(1 + \lambda^{-1})$. In addition the injectivity radius satisfies: $\mathrm{inj} \geqslant \pi/\lambda$.*

In [5, Theorem 4] it is shown that with the same assumptions the length of a closed geodesic c satisfies this estimate. Therefore, Proposition 1 follows from Proposition 3 which we are going to prove in this Note. Proposition 1 answers

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a question posed to the author by G. Paternain. Using results by Hofer, Wysocki and Zehnder [2] and the statement of Proposition 1 Harris and Paternain obtain the following geometric characterization of dynamically convex Finsler metrics on the 2-sphere:

Proposition 2. *(Harris–Paternain [1, Section 6].) Let F be a non-reversible Finsler metric on the 2-sphere with reversibility λ and flag curvature*

$$\left(1-\frac{1}{1+\lambda}\right)^2 < K \leqslant 1.$$

Then the Finsler metric is dynamically convex, *in particular there are either two geometrically distinct closed geodesics or there are infinitely many geometrically distinct ones.*

For existence results for closed geodesic of Finsler metrics on the 2-sphere we refer to the recent survey [4] by Long and to [7]. On the n-sphere S^n there is a 1-parameter family $F_\epsilon, \epsilon \in [0,1)$ of Finsler metrics (called *Katok metrics*) with the following properties: F_0 is the standard Riemannian metric, for every $\epsilon \in (0,1)$ the metric is a non-reversible Finsler metric of constant flag curvature 1, the reversibility is $\lambda = (1+\epsilon)/(1-\epsilon)$ and the shortest geodesic loop is a closed geodesic of length $\pi(1+\lambda^{-1})$. This shows that the estimate given in Proposition 1 is sharp. In addition the number of closed geodesics for $n=2$ is two if ϵ is irrational. It is an open problem whether there is a non-reversible Finsler metric on S^2 with a finite number $N > 2$ of geometrically distinct closed geodesics.

We use the following notations on a compact manifold M with Finsler metric F introduced in [5]: For points $p, q \in M$ let $\theta(p,q)$ be the minimal length of a piecewise differentiable curve $c:[0,1] \to M$ joining $p = c(0)$ and $q = c(1)$. For a non-reversible Finsler metric we have in general $\theta(p,q) \neq \theta(q,p)$, i.e. θ defines in general a non-symmetric metric on M. Therefore $\theta(p,q)$ equals the length of a minimal geodesic $c:[0,1] \to M$ joining $p = c(0)$ and $q = c(1)$, i.e. a geodesic with $L(c) = \theta(c(0), c(1))$. Then $d: M \times M \to \mathbb{R}; d(p,q) = (\theta(p,q) + \theta(q,p))/2$ defines a symmetric metric on M. For a point $p \in M$ and a unit vector $v \in T_pM; F(v) = 1$ let $c_v: \mathbb{R} \to M$ be the geodesic with $p = c(0); v = c'(0)$ and define $t_v := \sup\{t > 0; \theta(c(0), c(t)) = t\}$. Then $c_v(t_v)$ is a cut point of the point p and the cut locus $\mathrm{Cut}(p)$ of the point p is given by $\mathrm{Cut}(p) = \{c_v(t_v); v \in T_pM, F(v) = 1\}$.

We consider the following invariants and their relations: The *symmetrized injectivity radius* $d := \inf\{d(p,q); q \in \mathrm{Cut}(p)\}$, the length L of a shortest (nontrivial) closed geodesic and the length l of a shortest (nontrivial) geodesic loop.

Lemma 1. *Let (M, F) be a compact Finsler manifold, then the symmetrized injectivity radius d and the length l resp. L of a shortest non-trivial geodesic loop resp. closed geodesic satisfy: $2d \leqslant l \leqslant L$.*

Proof. Let $c:[0,l] \to M$ be a shortest geodesic loop parametrized by arc length with $p = c(0) = c(l)$. Let $q = c(t), t \in (0,l)$ be the cut point, i.e. $c|[0,t]$ is minimal. It follows that $l = L(c) \geqslant \theta(p,q) + \theta(q,p) = 2d(p,q) \geqslant 2d$. The inequality $L \geqslant l$ is obvious. □

The next ingredient in the proof of Proposition 3 is the following result:

Lemma 2. *(See [5, Lemma7].) Let (M, F) be a compact Finsler manifold with reversibility λ, symmetrized injectivity radius d and flag curvature $K \leqslant 1$. If $2d < \pi(1+\lambda^{-1})$ then the length l of a shortest non-trivial geodesic loop satisfies: $l = 2d$.*

With the help of these two lemmata we prove the following:

Proposition 3. *Let (M, F) be a compact manifold with Finsler metric F with reversibility λ and flag curvature $K \leqslant 1$. If the symmetrized injectivity radius d satisfies $2d < \pi(1+\lambda^{-1})$ then every shortest geodesic loop is a closed geodesic, hence $L = l = 2d$.*

Proof. Let $c:[0,l] \to M$ be a shortest geodesic loop parametrized by arc length with $c(0) = c(l) = p$. Let $q = c(t), t \in (0,l)$ be the cut point, i.e. $c|[0,t]$ is minimal. We assume that $l < \pi(1+\lambda^{-1})$. By Lemma 1 we obtain $2d < \pi(1+\lambda^{-1})$. Then we conclude from Lemma 2 that $l = L(c) = 2d$. Since

$$2d = l = L(c) \geqslant \theta(p,q) + \theta(q,p) = 2d(p,q) \tag{1}$$

and $q \in \mathrm{Cut}(p)$ it follows from the definition of the symmetrized injectivity radius d that equality holds in inequality (1). Therefore $c|[t,l]$ is a minimal geodesic joining q and p.

For sufficiently small $\epsilon > 0$ with $p_\epsilon = c(\epsilon)$ there is $t_\epsilon \in (t,1)$ such that $q_\epsilon = c(t_\epsilon) \in \mathrm{Cut}(p_\epsilon)$, i.e. the geodesic $c|[\epsilon, t_\epsilon]$ is minimal. We conclude from the triangle inequality:

$$\begin{aligned} 2d(p_\epsilon, q_\epsilon) &= \theta(p_\epsilon, q_\epsilon) + \theta(q_\epsilon, p_\epsilon) \leqslant \theta(p_\epsilon, q_\epsilon) + \theta(q_\epsilon, p) + \theta(p, p_\epsilon) \\ &= \theta(p,q) + \theta(q,p) = L(c) = 2d(p,q). \end{aligned}$$

From the definition of the symmetrized injectivity radius d it follows that actually equality holds, i.e. the geodesic loop is a closed geodesic. □

Proof of Proposition 1. We assume that $l < \pi(1+\lambda^{-1})$ and conclude from Lemmas 1 and 2: $2d = l < \pi(1+\lambda^{-1})$. Then Proposition 3 implies that $L = l = 2d < \pi(1+\lambda^{-1})$. But in [5, Theorem 4] it is shown that under the assumptions of Proposition 1 the length L of a shortest closed geodesic satisfies: $L \geqslant \pi(1+\lambda^{-1})$. Therefore we obtain a contradiction, i.e. $l \geqslant \pi(1+\lambda^{-1})$. □

Remark 1. Under the assumptions of Proposition 3 we have shown that for any point $p \in M$ with a cut point $q \in M$ satisfying $d(p,q) = d$ there is a shortest closed geodesic $c:[0,2d] \to M$ parametrized by arc length passing through p and q, i.e. $p = c(0) = c(2d)$; $q = c(t)$; $t \in (0,2d)$. Hence the restrictions $c_1 = c|[0,t]$ and $c_3 = c|[t,2d]$ are minimal geodesics. The cut point $q = c(t)$ is not a conjugate point since $t = \theta(p,q) < \pi$ and $K \leqslant 1$. This implies that there is another minimal geodesic $c_2:[0,t] \to M$ joining p and q. Therefore Proposition 3 excludes the second case discussed in [5, Remark 1] resp. [6, Lemma 9.7(b)].

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