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Number Theory

On the periodicity of an arithmetical function ☆

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Abstract

Let $k \geqslant 0$ be an integer. When studying the least common multiple of $k+1$ consecutive integers, Farhi introduced the arithmetical function g_k defined for any positive integer n by $g_k(n) := \frac{n(n+1)\cdots(n+k)}{\operatorname{lcm}(n,n+1,\ldots,n+k)}$. Farhi proved that g_k is periodic and $k!$ is a period of g_k. Meanwhile Farhi raised an open problem determining the smallest positive period of g_k. In this Note, we first show that $g_k(1) \mid g_k(n)$ for all positive integers n. Consequently, using this result, we show that for all positive integers k, $\operatorname{lcm}(1,2,\ldots,k)$ is a period of g_k, thus improving Farhi's result. ***To cite this article: S. Hong, Y. Yang, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur la périodicité d'une fonction arithmétique. Soit $k \geqslant 0$ un entier, en étudiant le plus petit commun multiple de $k+1$ entiers consécutifs Farhi a introduit la fonction arithmétique définie par $g_k(n) := \frac{n(n+1)\cdots(n+k)}{\operatorname{ppcm}(n,n+1,\ldots,n+k)}$ pour n entier positif. Farhi a démontré que g_k est périodique et que $k!$ en est une période. Dans le même temps Farhi a posé la question de déterminer la plus petite période de g_k. Dans cette Note, nous démontrons pour commencer $g_k(1) \mid g_k(n)$ pour tout entier positif n. Puis, utilisant ce résultat, nous montrons que $\operatorname{ppcm}(1,2,\ldots,k)$ est une période de g_k pour tout entier positif k, ce qui améliore le résultat de Farhi. ***Pour citer cet article : S. Hong, Y. Yang, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Il y a beaucoup de beaux théorèmes importants sur les progressions arithmétiques en théorie des nombres. Deux exemples sont le théorème de Dirichlet [1,7], et celui de Green–Tao [5]. Soit n un entier positif donné, Bachman et Kessler [2] et Myerson et Sander [11] ont étudié les propriétés de divisibilité du plus petit commun multiple de $1, 2, \ldots, n$, noté $\operatorname{ppcm}(1, 2, \ldots, n)$, tandis que Hong et Loewy [9] ont étudié le comportement asymptotique des valeurs propres des matrices de Smith définies par des progressions arithmétiques. Hanson [6] et Nair [12] ont obtenus des bornes supérieures et inférieures pour $\operatorname{ppcm}(1, 2, \ldots, n)$, respectivement. Fahri [3,4], a démontré des minorations non triviales pour le plus commun multiple de suites finies d'entiers. Hong et Feng [8] ont établi une minoration non

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triviale pour le plus petit commun multiple d'une progression arithmétique finie, qui confirme une conjecture de Farhi. Récemment, Hong et Yang [10] ont amélioré les minorations de Farhi et Hong et Feng.

D'un autre côté, Farhi [3,4], a étudié le plus petit commun multiple d'un nombre fini d'entiers consécutifs. Soit $k \geqslant 0$ un entier, il est démontré dans [3] et [4] que $\mathrm{ppcm}(n, n+1, \ldots, n+k)$ est divisible par $n\binom{n+k}{k}$ et aussi divise $n\binom{n+k}{k}\mathrm{ppcm}(\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{k})$. Farhi [3,4], a montré que la dernière divisibilité est une égalité lorsque $k! \mid (n+k+1)$. Ce faisant, Farhi introduit la fonction arithmétique g_k qui en tout entier positif n prend la valeur $g_k(n) := \frac{n(n+1)\cdots(n+k)}{\mathrm{ppcm}(n,n+1,\ldots,n+k)}$. Il démontre que la suite $\{g_k\}_{k=0}^{\infty}$ satisfait la formule récursive suivante :

$$g_k(n) = \mathrm{pgcd}\big(k!, (n+k)g_{k-1}(n)\big),$$

où $\mathrm{pgcd}(a, b)$ désigne le plus grand commun diviseur des entiers a et b. Utilisant cette formule, on peut facilement montrer (par récurrence sur k) que pour tout entier positif ou nul k la fonction g_k est périodique de période $k!$, c'est le résultat de Farhi [4]. Definissons P_k comme la plus petite période positive de g_k, alors le résultat de Farhi s'écrit $P_k \mid k!$ et Farhi [4] pose la question de déterminer la plus petite période positive de g_k.

Dans cette Note, nous étudions les périodes de g_k et nous montrons les deux théorèmes principaux suivants :

Théorème 0.1. *Pour tout entier positif ou nul k et tout entier positif n on a $g_k(1) \mid g_k(n)$.*

Théorème 0.2. *Pour tout entier positif ou nul k on a $P_k \mid \mathrm{ppcm}(1, 2, \ldots, k)$.*

Clairement, le Théorème 0.2 améliore le résultat de divisibilité de Farhi. Finalement, nous conjecturons $k \mid P_k$ pour tout entier $k \geqslant 1$ et $P_k = \mathrm{ppcm}(1, 2, \ldots, k)$ si $k+1$ est premier. De plus, nous conjecturons que $\frac{\mathrm{ppcm}(1,2,\ldots,k+1)}{k+1}$ divise P_k pour tout entier positif ou nul k. Évidemment, d'après le Théorème 0.2 les deux premières conjectures résultent de la dernière.

1. Introduction

There are many beautiful and important theorems about the arithmetic progression in number theory. Two examples are Dirichlet's theorem [1,7] and the Green–Tao theorem [5]. Let n be a given positive integer. Bachman and Kessler [2] and Myerson and Sander [11] investigated the divisibility properties of $\mathrm{lcm}(1, 2, \ldots, n)$ (i.e., the least common multiple of $1, 2, \ldots, n$) while Hong and Loewy [9] studied the asymptotic behavior of eigenvalues of Smith matrices defined on arithmetic progressions. Hanson [6] and Nair [12] found the upper bound and lower bound of $\mathrm{lcm}(1, 2, \ldots, n)$, respectively. Farhi [3,4] obtained some nontrivial lower bounds for the least common multiple of some finite sequences of integers. Hong and Feng published [8] a nontrivial lower bound for the least common multiple of finite arithmetic progressions which confirmed Farhi's conjecture. Recently, Hong and Yang [10] improved the lower bounds of Farhi, Hong and Feng.

On the other hand, Farhi [3,4] investigated the least common multiple of a finite number of consecutive integers. Let $k \geqslant 0$ be an integer. It was proved in [3,4] that $\mathrm{lcm}(n, n+1, \ldots, n+k)$ is divisible by $n\binom{n+k}{k}$ and also divides $n\binom{n+k}{k}\mathrm{lcm}(\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{k})$. Farhi [3,4] showed that the last equality holds if $k! \mid (n+k+1)$. In doing so, Farhi introduced the arithmetical function g_k which is defined for any positive integer n by $g_k(n) := \frac{n(n+1)\cdots(n+k)}{\mathrm{lcm}(n,n+1,\ldots,n+k)}$. Meanwhile, Farhi proved that the sequence $\{g_k\}_{k=0}^{\infty}$ satisfies the following recursive relation for all positive integers n:

$$g_k(n) = \gcd\big(k!, (n+k)g_{k-1}(n)\big), \tag{1}$$

where $\gcd(a, b)$ means the greatest common divisor of integers a and b. Using (1), we can easily show (by induction on k) that for any nonnegative integer k, the function g_k is periodic of period $k!$. This is a result due to Farhi [4]. Define P_k to be the smallest positive period of the function g_k. Then by Farhi's result, we have $P_k \mid k!$. In the meantime, Farhi [4] raised the open problem of determining the smallest positive period of g_k.

In this Note, we first show that $g_k(1) \mid g_k(n)$ for any nonnegative integer k and any positive integer n. Consequently, using this result, we show the second main result of this paper which states that $P_k \mid \mathrm{lcm}(1, 2, \ldots, k)$ for all positive integer k. That is, $\mathrm{lcm}(1, 2, \ldots, k)$ is a period of g_k. This improves Farhi's period. Finally we conjecture that $k \mid P_k$ for all integer $k \geqslant 1$ and $P_k = \mathrm{lcm}(1, 2, \ldots, k)$ if $k+1$ is a prime. Furthermore, we conjecture that $\frac{\mathrm{lcm}(1,2,\ldots,k+1)}{k+1}$ divides P_k for all nonnegative integers k.

Throughout this Note, let $\mathbf{Q}$ and $\mathbf{N}$ denote the field of rational numbers and the set of nonnegative integers. As usual, we let $v = v_p$ be the normalized p-adic valuation of $\mathbf{Q}$, i.e., $v(a) = b$ if $p^b \parallel a$. By $[x]$ and $\{x\}$ we denote the integer part and fractional part of a given real number x respectively. Then we have $0 \leqslant \{x\} < 1$ and $x = [x] + \{x\}$.

2. The main results and proofs

Throughout this Note, k denotes a fixed nonnegative integer. Let $n \geqslant 1$ be an integer. We first show that $g_k(1) \mid g_k(n)$. For this purpose, we need to prove several lemmas. Let $t \geqslant 0$ be an integer. Let p be any given prime. Define $S_p(n,t) := \{s \in \mathbf{N}\colon\ p^t \mid s, n \leqslant s \leqslant n+k\}$. Obviously, $S_p(n,t) \supseteq S_p(n,t+1)$ for any nonnegative integer t. We begin with the following result:

Lemma 2.1. *We have either* $|S_p(n,t)| = |S_p(1,t)|$ *or* $|S_p(n,t)| = |S_p(1,t)| + 1$.

Proof. Given l, N positive integers, we remark that in any set of l consecutive integers, there are exactly $[\frac{l-1}{N}] + 1$ integers divisible by N if the first integer n is divisible by N and $[\frac{l-1}{N}] + [\{\frac{l-1}{N}\} + \{\frac{n}{N}\}]$ otherwise. It follows that

$$\left|S_p(n,t)\right| = \begin{cases} \left[\dfrac{k}{p^t}\right] + \left[\left\{\dfrac{k}{p^t}\right\} + \left\{\dfrac{n}{p^t}\right\}\right] & \text{if } n \not\equiv 0 \ (\operatorname{mod} p^t), \\ \left[\dfrac{k}{p^t}\right] + 1 & \text{if } n \equiv 0 \ (\operatorname{mod} p^t). \end{cases}$$

So $|S_p(1,t)| = [\frac{k}{p^t}] + [\{\frac{k}{p^t}\} + \frac{1}{p^t}]$ if $t \geqslant 1$. But $[\{\frac{k}{p^t}\} + \{\frac{n}{p^t}\}] = 0$ or 1 for any integer $n \geqslant 1$ and $[\{\frac{k}{p^t}\} + \{\frac{n}{p^t}\}] \geqslant [\{\frac{k}{p^t}\} + \frac{1}{p^t}]$ if $p^t \nmid n$. Hence Lemma 2.1 is true if $t \geqslant 1$. Finally note that $|S_p(1,0)| = |S_p(n,0)| = k+1$. Therefore Lemma 2.1 holds if $t = 0$. The proof of Lemma 2.1 is complete. □

Define $m_p(n) := \max\{v_p(j) \mid n \leqslant j \leqslant n+k, j \in \mathbf{N}\}$. Then $m_p(n)$ is a nonnegative integer. We have the following three lemmas:

Lemma 2.2. *For any positive integer n, we have* $m_p(n) \geqslant m_p(1)$.

Proof. Because the set $\{n, n+1, \ldots, n+k\}$ has cardinality $k+1 \geqslant p^{m_p(1)}$, then it certainly contains at least an integer j which is a multiple of $p^{m_p(1)}$. Hence $m_p(n) \geqslant m_p(1)$, as required. Lemma 2.2 is proved. □

Lemma 2.3. *We have* $v_p(n(n+1)\cdots(n+k)) = \sum_{t=1}^{m_p(n)} |S_p(n,t)|$.

Proof. Let $t \geqslant 0$ be an integer. First note that $S_p(n,t) \supseteq S_p(n,t+1)$. Then $p^t \parallel x$ for any $x \in S_p(n,t) \setminus S_p(n,t+1)$. It infers that $v_p(x) = t$ for any $x \in S_p(n,t) \setminus S_p(n,t+1)$. By the definition of $m_p(n)$, we can see that $|S_p(n, m_p(n)+1)| = 0$. Thus we have

$$\begin{aligned} v_p\big(n(n+1)\cdots(n+k)\big) &= \sum_{i=n}^{n+k} v_p(i) = \sum_{x \in S_p(n,1)} v_p(x) \\ &= \sum_{t=1}^{m_p(n)} \sum_{x \in S_p(n,t)\setminus S_p(n,t+1)} v_p(x) = \sum_{t=1}^{m_p(n)} \big(\left|S_p(n,t)\right| - \left|S_p(n,t+1)\right|\big) \cdot t \\ &= \sum_{t=1}^{m_p(n)} \left|S_p(n,t)\right| \cdot t - \sum_{t=2}^{m_p(n)+1} \left|S_p(n,t)\right| \cdot t + \sum_{t=2}^{m_p(n)+1} \left|S_p(n,t)\right| = \sum_{t=1}^{m_p(n)} \left|S_p(n,t)\right| \end{aligned}$$

as desired. Thus Lemma 2.3 is proved. □

Lemma 2.4. *For any prime p, we have* $v_p\big(\binom{n+k}{k+1}\big) \geqslant m_p(n) - m_p(1)$.

Table 1
Some exact values of P_k

k	$\mathrm{lcm}(1,2,\ldots,k)$	P_k	k	$\mathrm{lcm}(1,2,\ldots,k)$	P_k
0	1	1	7	$420=2^2\cdot 3\cdot 5\cdot 7$	$105=3\cdot 5\cdot 7$
1	1	1	8	$840=2^3\cdot 3\cdot 5\cdot 7$	$280=2^3\cdot 5\cdot 7$
2	2	2	9	$2520=2^3\cdot 3^2\cdot 5\cdot 7$	$504=2^3\cdot 3^2\cdot 7$
3	$6=2\cdot 3$	3	10	$2520=2^3\cdot 3^2\cdot 5\cdot 7$	$2520=2^3\cdot 3^2\cdot 5\cdot 7$
4	$12=2^2\cdot 3$	$12=2^2\cdot 3$	11	$27\,720=2^3\cdot 3^2\cdot 5\cdot 7\cdot 11$	$27\,720=2^3\cdot 3^2\cdot 5\cdot 7\cdot 11$
5	$60=2^2\cdot 3\cdot 5$	$20=2^2\cdot 5$	12	$27\,720=2^3\cdot 3^2\cdot 5\cdot 7\cdot 11$	$27\,720=2^3\cdot 3^2\cdot 5\cdot 7\cdot 11$
6	$60=2^2\cdot 3\cdot 5$	$60=2^2\cdot 3\cdot 5$	13	$360\,360=2^3\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13$	$51\,480=2^3\cdot 3^2\cdot 5\cdot 11\cdot 13$

Proof. By Lemmas 2.1, 2.2 and 2.3, we have

$$v_p\left(\binom{n+k}{k+1}\right)=v_p\big(n(n+1)\cdots(n+k)\big)-v_p\big(1\cdot 2\cdots(k+1)\big)=\sum_{t=1}^{m_p(n)}\big|S_p(n,t)\big|-\sum_{t=1}^{m_p(1)}\big|S_p(1,t)\big|$$
$$=\sum_{t=1}^{m_p(1)}\big(\big|S_p(n,t)\big|-\big|S_p(1,t)\big|\big)+\sum_{t=m_p(1)+1}^{m_p(n)}\big|S_p(n,t)\big|\geqslant\sum_{t=m_p(1)+1}^{m_p(n)}\big|S_p(n,t)\big|. \tag{2}$$

Since $S_p(n,t)\neq\phi$ for all nonnegative numbers $t\leqslant m_p(n)$, it follows from (2) that

$$v_p\left(\binom{n+k}{k+1}\right)\geqslant\sum_{t=m_p(1)+1}^{m_p(n)}1=m_p(n)-m_p(1).$$

This completes the proof of Lemma 2.4. □

We can now prove the first main result of this Note:

Theorem 2.1. *For any given nonnegative integer k and any positive integer n, we have $g_k(1)\mid g_k(n)$.*

Proof. First we have

$$\frac{g_k(n)}{g_k(1)}=\frac{n\cdot(n+1)\cdots(n+k)}{1\cdot 2\cdots(k+1)}\cdot\frac{\mathrm{lcm}(1,2,\ldots,k+1)}{\mathrm{lcm}(n,n+1,\ldots,n+k)}=\binom{n+k}{k+1}\Big/\frac{\mathrm{lcm}(n,n+1,\ldots,n+k)}{\mathrm{lcm}(1,2,\ldots,k+1)}. \tag{3}$$

Notice that for any prime number p and any $n\in\mathbf{N}$, we have $v_p(\mathrm{lcm}(n,n+1,\ldots,n+k))=m_p(n)$. Then we can deduce, by using (3) and Lemma 2.4, that for any prime number p, we have $v_p(\frac{g_k(n)}{g_k(1)})\geqslant 0$, which concludes that $\frac{g_k(n)}{g_k(1)}$ is an integer, that is, $g_k(1)$ divides $g_k(n)$. So Theorem 2.1 is proved. □

Consequently, we give the second main result of this Note:

Theorem 2.2. *For any given nonnegative integer k, we have $P_k\mid\mathrm{lcm}(1,2,\ldots,k)$.*

Proof. The proof is using induction on k. For the cases that $k=0,1$, the assertion is trivial.

Assume that Theorem 2.2 is true for the $k\geqslant 1$ case. In what follows we show that Theorem 2.2 is true for the $k+1$ case. By the induction hypothesis, we have $P_k\mid\frac{k!}{g_{k-1}(1)}$. But $\mathrm{lcm}(1,2,\ldots,k)\mid\mathrm{lcm}(1,2,\ldots,k+1)$. Thus we get $\frac{k!}{g_{k-1}(1)}\mid\frac{(k+1)!}{g_k(1)}$. Hence $P_k\mid\frac{(k+1)!}{g_k(1)}$. So we have:

$$g_k\left(n+\frac{(k+1)!}{g_k(1)}\right)=g_k(n). \tag{4}$$

From (1) and (4) as well Theorem 2.1, we deduce that

$$\begin{aligned} g_{k+1}\big(n+\mathrm{lcm}(1,2,\ldots,k+1)\big) &= g_{k+1}\left(n+\frac{(k+1)!}{g_k(1)}\right) \\ &= \gcd\left((k+1)!, \left(n+k+1+\frac{(k+1)!}{g_k(1)}\right)g_k\left(n+\frac{(k+1)!}{g_k(1)}\right)\right) \\ &= \gcd\left((k+1)!, (n+k+1)g_k(n)+\frac{g_k(n)}{g_k(1)}\cdot(k+1)!\right) \\ &= \gcd\big((k+1)!, (n+k+1)g_k(n)\big) = g_{k+1}(n). \end{aligned}$$

It follows immediately that $P_{k+1} \mid \mathrm{lcm}(1,2,\ldots,k+1)$ as required. Hence Theorem 2.2 is proved. □

Remark. Since $k! > \mathrm{lcm}(1,2,\ldots,k)$ if $k \geqslant 4$, Theorem 2.2 is much better than Farhi's result. Using MATLAB and by some computations, we get the exact value of P_k when $k \leqslant 13$. We list them in Table 1. From it we find that $P_k = \mathrm{lcm}(1,2,\ldots,k)$ for $k = 0, 1, 2, 4, 6, 10, 11$ and 12. We guess that $k \mid P_k$ for all integer $k \geqslant 1$. We conjecture that $P_k = \mathrm{lcm}(1,2,\ldots,k)$ if $k+1$ is a prime. Furthermore, we conjecture that $\frac{\mathrm{lcm}(1,2,\ldots,k+1)}{k+1}$ divides P_k for all nonnegative integers k. Evidently, by Theorem 2.2 the previous two conjectures follow from the last conjecture. Finally, we point out that the question of determining P_k for all large integers k is still kept open.

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