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C. R. Acad. Sci. Paris, Ser. I 346 (2008) 745–748

COMPTES RENDUS MATHEMATIQUE

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Harmonic Analysis

Use of Hardy spaces and interpolation

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Received 15 May 2008; accepted 22 May 2008

Available online 20 June 2008

Presented by Yves Meyer

Abstract

We want to describe an abstract construction of Hardy spaces H^1 using an atomic decomposition and then we describe the use of these spaces in a point of view of interpolation. Mainly, we look for weakest assumptions to obtain an interpolation result between these Hardy spaces and Lebesgue spaces. ***To cite this article: F. Bernicot, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Utilisation des espaces de Hardy et interpolation. Nous présentons une construction abstraite d'espaces de Hardy H^1 par une décomposition atomique et nous décrivons l'utilisation de ces espaces avec pour but de les interpoler. Nous donnerons alors des hypothèses les plus faibles pour obtenir un résultat d'interpolation entre ces « nouveaux » espaces de Hardy et les espaces de Lebesgue. ***Pour citer cet article : F. Bernicot, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

In this Note, the author wants to improve the understanding of the use of Hardy spaces aiming to obtain continuities in Lebesgue spaces. Let us describe one of the main interest of Hardy spaces in the euclidean case. Assume that we have a linear operator T bounded on $L^{p_0}(\mathbb{R}^n)$ for an exponent $p_0 > 1$. If in addition T is bounded from the "classical" Hardy space $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, then by interpolation we deduce that T is $L^p(\mathbb{R}^n)$-bounded for all exponent $p \in (1, p_0)$.

We are very interested by this use of Hardy spaces and we want to describe the most abstract framework to compute this kind of argument. For ten years, many people have described in numerous papers some Hardy spaces, adapted to certain operators, by using maximal functions or atomic and molecular decompositions. It appears that the spaces obtained by atomic decompositions are smaller (strictly or not) than those defined by the corresponding maximal function.

That is why we shall define our Hardy spaces by an atomic structure and we will describe abstract assumptions to be able to interpolate our Hardy spaces with Lebesgue spaces. Specially, we want to emphasize that the study of the whole Hardy space is not useful for what we want. As we shall see, we have just to understand the action of an

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.05.009

operator on the set of atoms to obtain continuities in Lebesgue spaces. The structure of the whole Hardy space seems to be more useful to describe the dual space for example.

2. Definitions of Hardy spaces

Let (X, d, μ) be a space of homogeneous type: d is a quasi-distance on the space X and μ a Borel measure satisfying the doubling property:

$$\exists A > 0,\ \exists \delta > 0,\ \forall x \in X,\ \forall r > 0,\ \forall t \geqslant 1, \quad \frac{\mu(B(x,tr))}{\mu(B(x,r))} \leqslant A t^{\delta}, \tag{1}$$

where $B(x,r)$ is the open ball with center $x \in X$ and radius $r > 0$. We write $L^p := L^p(X, d, \mu)$ for the Lebesgue spaces.

We now define the Hardy spaces using abstract "oscillation operators". Let $\beta \in (1, \infty]$ be a fixed exponent and $\mathbb{B} := (B_Q)_{Q \in \mathcal{Q}}$ be a collection of L^β-bounded linear operators, indexed by $\mathcal{Q}$ the collection of all the balls Q of the space X. We assume that these operators B_Q are uniformly bounded on L^β: there exists a constant $0 < A' < \infty$ so that:

$$\forall f \in L^\beta,\ \forall Q \text{ ball}, \quad \left\| B_Q(f) \right\|_{L^\beta} \leqslant A' \|f\|_{L^\beta}. \tag{2}$$

We define *atoms* by using the collection $\mathbb{B}$:

Definition 2.1. A function $m \in L^1_{\mathrm{loc}}$ is called an atom associated to a ball Q if there exists a real function f_Q supported on the ball Q such that $m = B_Q(f_Q)$, with

$$\left(\int_Q |f_Q|^\beta \, \mathrm{d}\mu \right)^{1/\beta} \leqslant \mu(Q)^{-1/\beta'}.$$

The functions f_Q in this definition are normalized in L^1. It is easy to check that $\|f_Q\|_{L^1} \lesssim 1$. In [3], the authors have defined the concept of *molecules* associated to the collection $\mathbb{B}$. Now we can define our abstract atomic Hardy spaces:

Definition 2.2. A measurable function h belongs to the atomic Hardy space H^1_{ato} if there exists a decomposition:

$$h = \sum_{i \in \mathbb{N}} \lambda_i m_i \quad \mu\text{-a.e.},$$

where for all i, m_i is an atom and $(\lambda_i)_i$ are real numbers satisfying

$$\sum_{i \in \mathbb{N}} |\lambda_i| < \infty.$$

We equip H^1_{ato} with the norm:

$$\|h\|_{H^1_{\mathrm{ato}}} := \inf_{h = \sum_{i \in \mathbb{N}} \lambda_i m_i} \sum_i |\lambda_i|.$$

Remark 1. We only ask that the decomposition

$$h(x) = \sum_{i \in \mathbb{N}} \lambda_i m_i(x)$$

is well defined for almost every $x \in X$. So the assumption is very weak and it is possible that the measurable function h does not belong to L^1_{loc}. It is not clear whether this abstract normed vector space is complete. Fortunately, this property is not required for our aim.

Example 1. By the atomic decomposition of the Coifman–Weiss space (see [5]), we know that we exactly regain this space $H^1_{\mathrm{CW}} = H^1_{\mathrm{ato}}$ with

$$B_Q(f)(x) = f(x)\mathbf{1}_Q(x) - \mu(Q)^{-1}\left(\int f\,\mathrm{d}\mu\right)\mathbf{1}_Q(x).$$

In addition in this case, it is well-known and interesting to remark that this space does not depend on $\beta \in (1, \infty)$. For other examples and comparisons with already studied Hardy spaces, we refer the reader to Section 3 of [3].

As we will see, we need to use smaller spaces:

Definition 2.3. According to the collection $\mathbb{B}$, we introduce the set $H^1_{F,\mathrm{ato}} \subset H^1_{\mathrm{ato}} \cap L^\beta$, given by the finite sums of atoms with the following norm

$$\|f\|_{H^1_{F,\mathrm{ato}}} := \inf_{f=\sum_i \lambda_i m_i} \sum_i |\lambda_i|.$$

We take the infimum over all the finite atomic decompositions.

Remark 2. The norms on the atomic space H^1_{ato} and on the finite atomic space $H^1_{F,\mathrm{ato}}$ may not be equivalent (see a counterexample of Y. Meyer for the Coifman–Weiss space in [10]).

The use of these smaller spaces is very convenient. As explained in [4] and [9], to check that an operator admits a continuous extension on the Hardy space H^1_{ato} is a technical problem and requires extra assumptions. As we will see, in order to obtain interpolation results, we only use boundedness on the finite space $H^1_{F,\mathrm{ato}}$, which is far more simple to be satisfied.

3. Result of interpolation between Hardy and Lebesgue spaces

To describe our main result, we need to require some assumptions on the Hardy spaces and need some definitions:

Definition 3.1. For $\sigma \in [1, \infty]$ we define the maximal operator:

$$\forall x \in X, \quad M_\sigma(f)(x) := \sup_{\substack{Q \text{ ball} \\ x \in Q}} \left(\frac{1}{\mu(Q)} \int_Q \left|f - B^*_Q(f)\right|^\sigma \mathrm{d}\mu\right)^{1/\sigma} \tag{3}$$

and a sharp maximal function adapted to our operators: for $s > 0$,

$$\forall x \in X, \quad M^\sharp_s(f)(x) := \sup_{\substack{Q \text{ ball} \\ x \in Q}} \left(\frac{1}{\mu(Q)} \int_Q \left|B^*_Q(f)\right|^s \mathrm{d}\mu\right)^{1/s}.$$

The use of this sharp maximal function appeared in [8] and [6]. We recall the standard "Hardy–Littlewood" maximal operator, defined by: for $s > 0$,

$$\forall x \in X, \quad M_{\mathrm{HL},s}(f)(x) := \sup_{\substack{Q \text{ ball} \\ x \in Q}} \left(\frac{1}{\mu(Q)} \int_Q |f|^s \mathrm{d}\mu\right)^{1/s}.$$

We now come to the main result:

Theorem 3.2. *Assume that for $\sigma \in (\beta', \infty]$ and $p_0 \in (\sigma', \beta]$ the maximal operator M_σ is bounded by $M_{\mathrm{HL},p_0'}$. Let T be an L^{p_0}-bounded linearizable operator which is continuous from $H^1_{F,\mathrm{ato}}$ (or H^1_{ato}) into L^1. Then for all $p \in (\sigma', p_0]$ there exists a constant $C = C(p)$ such that*:

$$\forall f \in L^{p_0} \cap L^p, \quad \left\|T(f)\right\|_{L^p} \leqslant C\|f\|_{L^p}.$$

Remark 3. (1) What is very interesting, is that we only use the fact that there is a constant C such that for all atom m of H^1_{ato},

$$\left\| T(m) \right\|_{L^1} \leqslant C.$$

This assumption is equivalent to the continuity from $H^1_{F,\mathrm{ato}}$ to L^1 and is weaker than the continuity from H^1_{ato} to L^1. We do not require that T admits a continuous extension on the whole Hardy space H^1_{ato}, which seems to be a difficult technical problem to be solved.

(2) We refer the reader to Definition V.1.20 of [7] for the concept of "linearizable" operators.

Proof. The proof is already written in details in [3] for $p_0 = \beta = 2$ and in [2] in the general case using real interpolation theory. We just want to recall the main arguments and deal only with a linear operator T. The assumption that T is continuous from $H^1_{F,\mathrm{ato}}$ to L^1 implies that

$$\left\| M^{\sharp}_{\beta'}(T^* f) \right\|_{L^\infty} \lesssim \|f\|_{L^\infty}.$$

We use T^* for the adjoint operator. Then using the above assumption and the L^{p_0} boundedness of T, we have

$$\left\| M^{\sharp}_{\beta'}(T^* f) \right\|_{L^{p_0'},\infty} \lesssim \left\| M_{\mathrm{HL},p_0'}(T^* f) \right\|_{L^{p_0'},\infty} \lesssim \|f\|_{L^{p_0'}}.$$

Now using real interpolation for the sublinear operator $M^{\sharp}_{\beta'}(T^*)$, we get that for all $p \in (1, p_0)$

$$\left\| M^{\sharp}_{\beta'}(T^* f) \right\|_{L^{p'}} \lesssim \|f\|_{L^{p'}}. \tag{4}$$

Then we use a "good lambda inequality" to compare the maximal functions $M_{\mathrm{HL},\beta'}$ and $M^{\sharp}_{\beta'}$. Using Theorem 3.1 of [1], we also obtain

$$\|h\|_{L^q} \lesssim \left\| M_{\mathrm{HL},\beta'}(h) \right\|_{L^q} \lesssim \left\| M^{\sharp}_{\beta'}(h) \right\|_{L^q} \tag{5}$$

for all functions $h \in L^q \cap L^{\beta'}$ and all exponent $q < \sigma$. Together (4) and (5) give us that T^* admits a continuous extension on L^q for $q \in (p_0', \sigma)$. The proof is completed by duality. $\square$

Remark 4. We can obtain a stronger result characterizing the intermediate spaces with real interpolation theory (see [2]). However these arguments seem to require a space of infinite measure $\mu(X) = \infty$ and the continuous embedding $H^1_{F,\mathrm{ato}} \hookrightarrow L^1$.

Example 2. In the case of the Coifman–Weiss space, we have seen in Example 1, that $H^1_{\mathrm{CW}} = H^1_{\mathrm{ato}}$ with a special choice for the operators B_Q and $\beta = \infty$. In this particular case, it is obvious to check that our maximal operator M_∞ is bounded by $M_{\mathrm{HL},1}$. We regain also the "classical result": H^1_{CW} can be interpolated with all the Lebesgue spaces L^p (with $1 < p \leqslant \infty$) to find the intermediate Lebesgue spaces.

We finish by referring the reader to [3] for a criterion of $H^1_{F,\mathrm{ato}}$-L^1 boundedness, a detailed study of these abstract Hardy spaces and to [2,3] for applications of these Hardy spaces and interpolation results.

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