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Group Theory

A well-ordering of dual braid monoids

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Abstract

Let B_n^{+*} denote the dual braid monoid on n strands, i.e., the submonoid of the braid group B_n consisting of the braids that can be expressed as positive words in the Birman–Ko–Lee generators. We introduce a new normal form on B_n^{+*}, which is based on expressing every braid of B_n^{+*} in terms of a certain finite sequence of braids of B_{n-1}^{+*}. We deduce an inductive characterization of the Dehornoy ordering of dual braid monoids, and explicitly compute the associated order types. ***To cite this article: J. Fromentin, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Bon ordre du monoïde de tresses dual. Soit B_n^{+*} le monoïde de tresses dual sur n brins, c'est-à-dire le sous-monoïde du groupe de tresses B_n formé par les tresses ayant une expression positive en les générateurs de Birman–Ko–Lee. Nous introduisons une nouvelle forme normale, dite cyclante, sur B_n^{+*}. Cette forme normale est basée sur une décomposition de chaque tresse de B_n^{+*} en termes d'une suite de tresses de B_{n-1}^{+*}. Nous en déduisons une caractérisation inductive de l'ordre de Dehornoy sur les monoïdes de tresses duaux, et calculons explicitement les types d'ordre associés. ***Pour citer cet article : J. Fromentin, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

On sait depuis [6] et [10] que le groupe de tresses B_n admet un ordre compatible avec la multiplication à gauche, dont la restriction au monoïde de tresses positives B_n est un bon ordre [4,9]. Cet ordre, usuellement appelé ordre de Dehornoy, a maintenant reçu de nombreuses définitions reflètant différentes approches possibles [7]. Dans cette Note, nous considérons la version dite supérieure de l'ordre des tresses, dans laquelle β est déclarée plus petite que β', noté $\beta < \beta'$, si la tresse $\beta^{-1}\beta'$ admet une expression dans laquelle le générateur σ_i de plus grand indice n'apparaît que positivement, auquel cas elle est dite σ-positive.

Nous étudions la restriction de l'ordre des tresses aux monoïdes de tresses duaux. Le monoïde de tresses dual B_n^{+*} est défini comme le sous-monoïde de B_n engendré par la famille des tresses $\mathtt{a}_{i,j}$ avec $1 \leqslant i < j \leqslant n$, où $\mathtt{a}_{i,j}$ désigne la tresse $\sigma_i \cdots \sigma_{j-2}\sigma_{j-1}\sigma_{j-2}^{-1} \cdots \sigma_i^{-1}$ (les σ_i sont les générateurs standards d'Artin [1]). Birman, Ko et Lee ont

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établi dans [3] que B_n^{+*} admet la présentation (1) relativement aux générateurs $\mathtt{a}_{i,j}$, ainsi qu'une structure de Garside associée à l'élément de Garside δ_n défini par $\delta_n = \sigma_1\sigma_2\cdots\sigma_{n-1}$, les éléments simples correspondant aux partitions sans croisements de n [2]. On note ϕ_n l'automorphisme de B_n^{+*} associé à la conjugaison par δ_n, qui envoie β sur $\delta_n\beta\delta_n^{-1}$.

Suivant l'approche de [5], nous associons à chaque tresse de B_n^{+*} une suite de tresses de B_{n-1}^{+*}, appelée ϕ_n-éclatement de β, qui spécifie β complètement (voir Fig. 2).

Proposition 0.1. *Soit β une tresse non triviale de B_n^{+*}. Alors il existe une unique suite $(\beta_p,\ldots,\beta_1)$ de B_{n-1}^{+*}, appelée* ϕ_n-éclatement de β, *vérifiant $\beta_p \neq 1$, $\beta = \phi_n^{p-1}(\beta_p)\cdot\cdots\cdot\phi_n(\beta_2)\cdot\beta_1$ et telle que, pour chaque r, la tresse β_r est le plus grand diviseur à droite de $\phi_n^{p-r}(\beta_p)\cdot\cdots\cdot\beta_r$ dans B_{n-1}^{+*}.*

Comme toute tresse de B_2^{+*} est représentée par un unique mot en $\mathtt{a}_{1,2}$, on obtient récursivement, à l'aide du ϕ_n-éclatement, une forme normale sur B_n^{+*} pour tout $n \geqslant 2$, appelée *forme normale cyclante* :

Définition 0.2.

(i) Pour β dans B_2^{+*}, la *forme normale cyclante* de β est définie comme étant l'unique mot $\mathtt{a}_{1,2}^e$ représentant β.

(ii) Pour $n \geqslant 3$ et β dans B_n^{+*}, la *forme normale cyclante* de β est définie comme étant le mot $\phi_n^{p-1}(w_p)\cdots w_1$ sur l'alphabet $\{\mathtt{a}_{i,j} \mid 1 \leqslant i < j \leqslant n\}$, où $(\beta_p,\ldots,\beta_1)$ est le ϕ_n-éclatement de β et, pour chaque r, le mot w_r est la forme normale cyclante de β_r.

Le résultat principal est que la restriction de l'ordre des tresses à B_n^{+*} admet une construction simple à partir de sa restriction à B_{n-1}^{+*}.

Théorème 0.3. *Soient β et β' deux tresses de B_n^{+*}. Alors la relation $\beta < \beta'$ est vraie si et seulement si le ϕ_n-éclatement de β est plus petit que le ϕ_n-éclatement de β' pour l'extension* `ShortLex` *de l'ordre sur B_{n-1}^{+*}.*

On rappelle que, si $(A, \prec)$ est un ensemble ordonné, une suite finie S d'éléments de A est dite `ShortLex` plus petite qu'une autre suite finie S' si la longueur de S est strictement plus petite que celle de S', ou alors si les longueurs de S et S' sont égales et S est $\prec$-lexicographiquement plus petite que S', c'est-à-dire lorsque les deux suites sont lues en partant de la gauche, le premier terme dans S qui ne coïncide pas avec sa contrepartie dans S' est plus petit pour l'ordre $\prec$.

Afin de démontrer le Théorème 0.3, nous définissons un nouvel ordre $<^*$ sur B_n^{+*} vérifiant de manière évidente la conclusion du théorème. Nous calculons un représentant *σ-positif* du quotient $\beta^{-1}\beta'$ lorsque les tresses β et β' de B_n^{+*} satisfont $\beta <^* \beta'$. Le calcul est direct et effectif, donc, en particulier, on en déduit un algorithme permettant de trouver un représentant σ-positif. Décrire le calcul et prouver qu'il donne le résultat souhaité est délicat, c'est pourquoi nous le décomposons en plusieurs étapes. Pour cela, nous introduisons une suite de tresses $\hat{\delta}_{n,r}$ jouant le rôle de séparateurs pour l'ordre $<^*$ (voir Fig. 3). Nous démontrons alors le Théorème 0.3 lorsque la tresse β ou bien la tresse β' est un séparateur $\hat{\delta}_{n,r}$ et en déduisons le cas général.

Un corollaire immédiat du Théorème 0.3 montre que le type d'ordre de $(B_n^{+*}, <)$ est l'ordinal $\omega^{\omega^{n-2}}$.

1. Introduction

It is known since [6] and [10] that Artin's braid group B_n is left-orderable, by an ordering whose restriction to the positive braid monoid B_n is a well-ordering. This ordering of B_n, which is usually called the Dehornoy ordering, has now received a lot of alternative constructions that reflect the many different possible approaches [7]. The aim of this Note is to study the restriction of the braid ordering to the dual braid monoids.

We use σ_i for the standard Artin generators of braid groups [1].

Definition 1.1. For $1 \leqslant i < j$, we put $\mathtt{a}_{i,j} = \sigma_i\cdots\sigma_{j-2}\sigma_{j-1}\sigma_{j-2}^{-1}\cdots\sigma_i^{-1}$, and, for $n \geqslant 2$, we define the *dual braid monoid* B_n^{+*} to be the submonoid of B_n generated by all the elements $\mathtt{a}_{i,j}$ with $1 \leqslant i < j \leqslant n$.

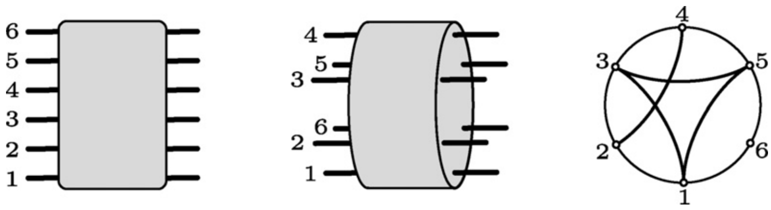

Fig. 1. Rolling up the usual diagram helps up to see the symmetries of the $a_{i,j}$'s. On the resulting cylinder, $a_{i,j}$ naturally corresponds to the chord connecting the vertices i and j.

Fig. 1. Enrouler le diagramme usuel permet de voir les symétries des $a_{i,j}$. Sur le cylindre obtenu, $a_{i,j}$ correspond naturellement à la corde reliant les points i et j.

Note that σ_i coincides with $a_{i,i+1}$ and, therefore, the standard braid monoid B_n^+ is included in B_n^{+*} for each n. For $n \geqslant 3$, the inclusion is proper, as the braid $a_{1,3}$ belongs to B_n^{+*}, but not to B_n^+.

We write $[i, j]$ for the interval $\{i, \dots, j\}$ of $\mathbb{N}$, and we say that $[i', j']$ is *nested* in $[i, j]$ if we have $i < i' < j' < j$.

Proposition 1.1 *(Birman, Ko, Lee [3]). The braid monoid B_n^{+*} is presented by the generators $a_{i,j}$ subject to the relations*

$$\begin{aligned} &a_{i,j}a_{i',j'} = a_{i',j'}a_{i,j} \quad \text{for } [i,j] \text{ and } [i',j'] \text{ disjoint or nested},\\ &a_{i,j}a_{j,k} = a_{j,k}a_{i,k} = a_{i,k}a_{i,j} \quad \text{for } 1 \leqslant i < j < k \leqslant n. \end{aligned} \tag{1}$$

Note that our generators $a_{i,j}$ are not those exactly considered in [3], but their image under the *flip*-automorphism of B_n, which comes from the Garside structure of B_n^+ [8].

Moreover, the results of [3] show that B_n^{+*} is what is now called a Garside monoid with respect to the Garside element δ_n defined by $\delta_n = \sigma_1\sigma_2\cdots\sigma_{n-1}$, where simple elements correspond to non-crossing partitions of n [2]. We denote by ϕ_n the automorphism of B_n^{+*} that maps β to $\delta_n\beta\delta_n^{-1}$. Then we have $\phi_n(a_{i,j}) = a_{i+1,j+1}$ for $j \leqslant n-1$ and $\phi_n(a_{i,n}) = a_{1,i+1}$, so, in chord representation it acts as a rotation—see Fig. 1. We shall also use ϕ_n for the alphabetical homomorphism on words in the letters $a_{i,j}$ defined by the above formulas.

2. The ϕ_n-splitting of a dual braid, and the cycling normal form

The first step in our analysis of the dual braid monoid B_n^{+*} consists in associating with every n-strand dual braid a finite sequence of $(n-1)$-strand dual braids that determines it completely. This in turn is achieved by iterating the following simple operation:

Lemma 2.1. *(See [5].) Assume $n \geqslant 3$. Every braid β of B_n^{+*} admits a maximal right divisor β_1 lying in B_{n-1}^{+*}. For such a β_1, the maximal right divisor of $\beta\beta_1^{-1}$ lying in B_{n-1}^{+*} is trivial.*

The braid β_1 given in Lemma 2.1 is called the B_{n-1}^{+*}-*tail* of β.

Proposition 2.1. *Assume that β is a non-trivial braid of B_n^{+*}. Then there exists a unique sequence $(\beta_p, \dots, \beta_1)$ in B_{n-1}^{+*} satisfying $\beta_p \neq 1$, $\beta = \phi_n^{p-1}(\beta_p) \cdot \cdots \cdot \phi_n(\beta_2) \cdot \beta_1$ and such that for each r, the braid β_r is the B_{n-1}^{+*}-tail of $\phi_n^{p-r}(\beta_p) \cdot \cdots \cdot \beta_r$.*

The unique sequence $(\beta_p, \dots, \beta_1)$ of braids introduced in Proposition 2.1 is called the ϕ_n-*splitting of* β, and its length is called the n-*breadth* of β—see Fig. 2.

Each braid of B_2^{+*} is a power of $a_{1,2}$, hence it is represented by a unique word of the form $a_{1,2}^e$. Using the ϕ_n-splitting recursively, we deduce a distinguished representative word for each braid of B_n^{+*}.

Definition 2.2. (i) For β in B_2^{+*}, the *cycling normal form* of β is defined to be the unique word $a_{1,2}^e$ that represents β.

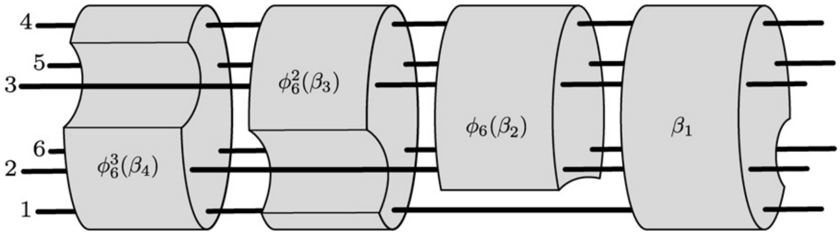

Fig. 2. The ϕ_n-splitting of a braid of B_n^{+*}. Starting from the right, we take the maximal right divisor that keeps the last strand unbraided, then take the maximal right divisor of the remainder that keeps the first strand unbraided, and so on turning by $2\pi/n$ counterclockwise at each step.

Fig. 2. Le ϕ_n-éclatement d'une tresse de B_n^{+*}. Partant de la droite, on prend le plus grand diviseur à droite laissant le dernier brin non tressé, puis on prend le plus grand diviseur à droite du reste laissant le premier brin non tressé, et ainsi de suite en tournant de $2\pi/n$ dans le sens des aiguilles d'une montre à chaque étape.

(ii) For $n \geqslant 3$ and β in B_n^{+*}, the *cycling normal form* of β is defined to be the word $\phi_n^{p-1}(w_p) \cdots \phi_n(w_2) w_1$, where $(\beta_p, \ldots, \beta_1)$ is the ϕ_n-splitting of β and, for each r, the word w_r is the cycling normal form of β_r.

3. The main result

We recall the definition of the braid ordering in terms of σ-positive words; here $<$ will refer to the upper version of that ordering, in which one considers the generator σ_i with highest index, rather than the one with lower index.

Definition 3.1.

(i) A braid word (in the letters σ_i) is said to be *σ_i-positive* if w contains at least one σ_i, no σ_i^{-1}, and no letter $\sigma_j^{\pm 1}$ with $j > i$.
(ii) A braid β is said to be *σ_i-positive* if there exists at least one σ_i-positive word representing β.
(iii) If β, β' are braids, we declare that $\beta < \beta'$ is true if the quotient $\beta^{-1}\beta'$ is σ-positive, i.e., σ_i-positive for some i.

Proposition 3.1. *(See [6].) For each n, the relation $<$ is a linear ordering of B_n that is invariant under left multiplication.*

By construction, each generator $a_{i,j}$ is σ-positive, and, therefore, every braid in B_n^{+*} except 1 is larger than 1 with respect to the braid ordering.

Our main result is the following structural result for the restriction of the braid ordering to the dual braid monoid B_n^{+*}:

Theorem 3.2. *The ordering $<$ of the dual braid monoid B_n^{+*} admits the following recursive characterization: for β, β' in B_n^{+*}, the relation $\beta < \beta'$ holds if and only if, denoting by $(\beta_p, \ldots, \beta_1)$ and $(\beta'_{p'}, \ldots, \beta'_1)$ the ϕ_n-splittings of β and β' respectively, the sequence $(\beta_p, \ldots, \beta_1)$ is smaller than the sequence $(\beta'_{p'}, \ldots, \beta'_1)$ with respect to the* `ShortLex`*-extension of the ordering of B_{n-1}^{+*}.*

We recall that, if $(A, \prec)$ is an ordered set, a finite sequence S in A is called `ShortLex`-smaller than another finite sequence S' if the length of S is smaller than that of S', or if both lengths are equal and S is lexicographically $\prec$-smaller than S', i.e., when both sequences are read starting from the left, the first entry in S that does not coincide with its counterpart in S' is $\prec$-smaller.

A direct consequence of Theorem 3.2 is the following result, which strictly refines a former non-effective result by Laver [10] based on Higman's Lemma [9] and provides the effective determination of the order type of the well-ordering:

Corollary 3.3. *The ordering $<$ of B_n^{+*} is a well-ordering, and its order type is the ordinal $\omega^{\omega^{n-2}}$.*

Proof. Let λ_n be the order type of the restriction of the braid ordering to B_n^{+*}. The ordinal type of $<^*$ on B_2^{+*} is the order type of the standard ordering of natural numbers, namely ω, so we have $\lambda_2 = \omega$. Then, it is standard that, if $(X, <)$ is a well-ordering of ordinal type λ, then the ShortLex-extension of $<$ on the set of all finite sequences in X is a well-ordering of ordinal type λ^ω. So, an immediate induction gives $\lambda_n \leqslant \omega^{\omega^{n-2}}$ for each $n \geqslant 2$. On the other hand, it was shown in [4]—or in [7]—that the ordinal type of the restriction of $<$ to B_n is $\omega^{\omega^{n-2}}$. As B_n is included in B_n^{+*}, we deduce $\lambda_n \geqslant \omega^{\omega^{n-2}}$, and, finally, we obtain $\lambda_n = \omega^{\omega^{n-2}}$. □

4. A linear ordering on B_n^{+*}

In order to prove Theorem 3.2, we use an indirect approach: we first introduce an ordering of B_n^{+*}, denoted $<^*$, that obeys the recursive rule of Theorem 3.2. We investigate this auxiliary ordering, and eventually deduce that it coincides with the original braid ordering, so that the latter obeys the expected recursive rule.

Definition 4.1. For $n \geqslant 2$, we recursively define $<_n^*$ on B_n^{+*} as follows:

(i) For β, β' in B_2^{+*}, we declare that $\beta <_2^* \beta'$ is true if we have $\beta = \mathtt{a}_{1,2}^{p}$ and $\beta' = \mathtt{a}_{1,2}^{p'}$ with $p < p'$;
(ii) For β, β' in B_n^{+*} with $n \geqslant 3$, we declare that $\beta <_n^* \beta'$ is true if the ϕ_n-splitting of β is smaller than that of β' for the ShortLex-extension of $<_{n-1}^*$.

Proposition 4.1. *For $n \geqslant 2$, the relation $<_n^*$ is a well-ordering of B_n^{+*}*

Proof. The ordered monoid $(B_2^{+*}, <_2^*)$ is isomorphic to $\mathbb{N}$ with the usual ordering, which is a well-ordering. As the ShortLex-extension of a well-ordering is also a well-ordering, we deduce using induction on n that $<_n^*$ is a well-ordering. □

For $n \geqslant 3$, the monoid B_{n-1}^{+*} turns out to be the initial segment of $(B_n^{+*}, <_n^*)$ determined by $\mathtt{a}_{n-1,n}$, i.e., we have $B_{n-1}^{+*} = \{\beta \in B_n^{+*} \mid \beta <_n^* \mathtt{a}_{n-1,n}\}$. So $<_{n-1}^*$ is the restriction of $<_n^*$ to B_{n-1}^{+*}, and we can skip the subscript and write $<^*$ instead of $<_n^*$.

At this point, we have two a priori unrelated linear orderings of the dual braid monoid B_n^{+*}, namely the standard (Dehornoy) ordering and the ordering $<^*$, which is a well-ordering (but is not known so far to be invariant under left-multiplication). According to the scheme above, our main technical result will be:

Proposition 4.2. *For each n, the orderings $<^*$ and $<$ coincide on B_n^{+*}.*

Because both $<$ and $<^*$ are linear orderings, it is actually enough to prove:

Proposition 4.3. *For all braids β, β' in B_n^{+*}, the relation $\beta <^* \beta'$ implies $\beta < \beta'$.*

This result will be established by considering two braids β, β' in B_n^{+*} satisfying $\beta <^* \beta'$ and computing an explicit σ-positive word that represents the quotient $\beta^{-1}\beta'$. The computation—which is effective and gives an actual algorithm—is based on necessary properties satisfied by words that are normal in the sense of Definition 2.2.

5. The principle of the argument

Describing the computation and proving its correctness is a rather delicate process, and we shall split the computation into several steps. Our problem is to prove

$$\forall \beta, \beta' \in B_n^{+*} \quad \beta <^* \beta' \quad \text{implies} \quad \beta < \beta'. \tag{2}$$

The first step consists in replacing the initial problem that involves two arbitrary braids β, β' with two problems, each of which only involves one braid. To this end, we introduce a sequence of specific braids $\hat{\delta}_{n,r}$ that are closely connected with the powers of the Garside braids δ_n and that play the role of separators in the $<^*$-ordering—see Fig. 3.

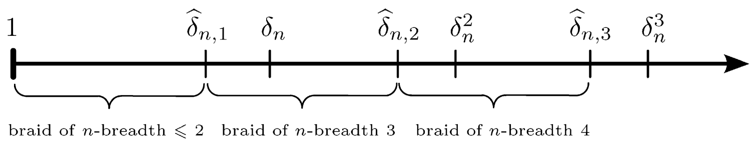

Fig. 3. The braid $\hat{\delta}_{n,r}$ as separator in $(B_n^{+*}, <^*)$—hence in $(B_n^{+*}, <)$ as well once Theorem 3.2 is proved.

Fig. 3. La tresse $\hat{\delta}_{n,r}$ comme séparateur dans $(B_n^{+*}, <^*)$—donc dans $(B_n^{+*}, <)$ une fois que le Théorème 3.2 est prouvé.

Definition 5.1. For $n \geqslant 3$ and $r \geqslant 1$, we put $\hat{\delta}_{n,r} = \delta_n^r \delta_{n-1}^{-r}$.

It is easily seen that the ϕ_n-splitting of $\hat{\delta}_{n,r}$ is the sequence $(\mathtt{a}_{n-2,n-1}, \ldots, \mathtt{a}_{n-2,n-1}, 1, 1)$, with $\mathtt{a}_{n-2,n-1}$ repeated r times. So $\hat{\delta}_{n,r}$ has n-breadth $r+2$, and it is the $<^*$-smallest braid in B_n^{+*} that has this breadth. Hence $\hat{\delta}_{n,r}$ is the $<^*$-least upper bound for all braids with n-breadth at most $r+1$.

Let us assume that β and β' belong to B_n^{+*} and $\beta <^* \beta'$ holds. We wish to prove (2), i.e., we wish to prove that $\beta^{-1}\beta'$ is σ-positive.

Assume that the n-breadth of β' is $r+2$—the special cases when β' has small breadth are treated separately. The hypothesis $\beta <^* \beta'$ implies that the n-breadth of β is at most $r+2$.

If the n-breadth of β is strictly smaller than $r+2$, an easy computation shows that $\beta^{-1}\hat{\delta}_{n,r}$ is σ_{n-1}-positive, and a not so easy computation using induction on r together with necessary properties satisfied by cycling normal words show that $\hat{\delta}_{n,r}^{-1}\beta'$ is σ_{n-1}-positive or σ_{n-1}-free, implying that $\beta^{-1}\beta'$ is σ_{n-1}-positive.

If the n-breadth of β equals $r+2$, we consider the leftmost entries in the ϕ_n-splittings of β and β' that do not coincide. Appealing to the result for $n-1$, we find a σ-positive expression for their quotient, and then deduce from explicit computations a σ-positive expression for the quotient $\beta^{-1}\beta'$.

References

[1] E. Artin, Theory of braids, Ann. of Math. 48 (1947) 101–126.
[2] D. Bessis, The dual braid monoid, Ann. Sci. Norm. Sup. 36 (2003) 647–683.
[3] J. Birman, K.H. Ko, S.J. Lee, A new approach to the word problem in the braid groups, Adv. in Math. 139 (1998) 322–353.
[4] S. Burckel, The wellordering on positive braids, J. Pure Appl. Algebra 120 (1) (1994) 1–17.
[5] P. Dehornoy, Alternating normal forms for braids and locally Garside monoids, arXiv: math.GR/0702592.
[6] P. Dehornoy, Braid groups and left distributive operations, Trans. Amer. Math. Soc. 345 (2) (1994) 293–304.
[7] P. Dehornoy, I. Dynnikov, D. Rolfsen, B. Wiest, Ordering Braids, Mathematical Surveys and Monographs, Amer. Math. Soc., in press.
[8] F.A. Garside, The braid group and other groups, Quart. J. Math. Oxford 20 (1969) 235–254.
[9] G. Higman, Ordering by divisibility in abstract algebras, Proc. London Math. Soc. 2 (1952) 326–336.
[10] R. Laver, Braid group actions on left distributive structures and well-orderings in the braid group, J. Pure Appl. Algebra 108 (1) (1996) 81–98.