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Dynamical Systems

Persistence of stratifications of normally expanded laminations

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Abstract

We introduce here the concept of stratification of laminations. We explain also a sufficient condition which provides the C^1-persistence of a stratification of laminations preserved by a C^1-endomorphism of a manifold. We present various applications of this result. ***To cite this article: P. Berger, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Persistance des stratifications de laminations normalement dilatées. On introduit ici la notion de stratification de laminations. On décrit aussi une condition suffisante assurant la persistance des stratifications de laminations préservées par un C^1-endomorphisme d'une variété. On présente des applications variées de ce résultat. ***Pour citer cet article : P. Berger, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Rappelons qu'une *lamination* (de dimension d) est un espace séparable localement modelé (via un atlas de cartes compatibles) sur le produit de $\mathbb{R}^d$ avec un espace localement compact (voir [1], p. 32). Un *plongement* d'une lamination $\mathcal{L}$ dans une variété M est un homéomorphisme i de $\mathcal{L}$ sur son image dans M, dont la différentielle le long des feuilles existe, est injective et varie continûment sur $\mathcal{L}$. Une lamination $\mathcal{L}$ plongée dans M est *préservée* par un C^1-endomorphisme f de M si chaque feuille de $\mathcal{L}$ est envoyée par f dans une feuille de $\mathcal{L}$. Une telle lamination est *persistante* si, pour toute C^1-perturbation f' de f, il existe un plongement i' de $\mathcal{L}$ dans M proche de i tel que f' préserve le plongement de $\mathcal{L}$ par i' et tel que les dynamiques induites par f et f' sur l'espace des feuilles de $\mathcal{L}$ sont les mêmes.

Dans le cadre des difféomorphismes, Hirsch–Pugh–Shub [3] ont montré la persistance des laminations normalement hyperboliques et expansives par plaques. De façon duale, on a montré :

Théorème 0.1. *Soit $\mathcal{L}$ une lamination compacte plongée dans une variété riemannienne M. Soit f un C^1-endomorphisme de M, dilatant normalement $\mathcal{L}$ et étant expansif par plaques. La lamination $\mathcal{L}$ est alors persistante.*

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Suivant Mather [4], un *espace stratifié* est un espace métrique compact A muni d'une partition finie Σ en parties localement fermées (les *strates*) qui vérifient la condition de frontière suivante :

$$\forall (X, Y) \in \Sigma^2, \quad \mathrm{adh}(X) \cap Y \neq \emptyset \Rightarrow \mathrm{adh}(X) \supset Y. \quad \text{La strate } X \text{ est alors dite } \textit{incidente} \text{ à } Y.$$

Un *espace stratifié laminaire* est un espace stratifié (A, Σ) dont les strates sont munies d'une structure de lamination, tel que si X est incidente à Y alors la dimension de X est supérieure ou égale à celle de Y. On dit aussi que Σ est une *stratification de laminations*. Un *plongement* de (A, Σ) dans une variété M est un homéomorphisme de A sur son image, dont la restriction à chaque strate est un plongement de lamination. Un C^1-endomorphisme f de M *préserve* un espace stratifié laminaire (A, Σ) plongée par i dans M s'il préserve chaque strate (munie de sa structure). La stratification de laminations Σ est *persistante* si toute C^1-perturbation f' de f préserve un plongement i' de (A, Σ) dans M proche de i tel que les dynamiques induites par f et f' sur l'espace des feuilles de chaque strate de Σ soient les mêmes.

On va donner des conditions suffisantes pour qu'une stratification de laminations soit persistante. On a montré qu'il existe des stratifications normalement dilatées qui ne sont pas persistantes. Des conditions supplémentaires sont donc requises pour assurer leur persistance. Pour cela, on définit la structure de *treillis* (*de laminations*) sur un espace stratifié.

Une structure de treillis $\mathcal{T}$ sur un espace stratifié laminaire (A, Σ) est la donnée, pour chaque strate $X \in \Sigma$, d'une lamination $\mathcal{L}_X$ telle que :

- la lamination $\mathcal{L}_X$ couvre un voisinage ouvert de X et compte parmis ses feuilles toutes celles de X,
- si Y est incidente à X, les plaques de $\mathcal{L}_Y$ incluses dans $\mathcal{L}_X$ sont C^1-feuilletées par des plaques de $\mathcal{L}_X$, et ce feuilletage varie continûment transversalement aux plaques de $\mathcal{L}_Y$.

Un plongement de (A, Σ) dans une variété M est $\mathcal{T}$*-contrôlé* si la restriction de i à chaque lamination $\mathcal{L}_X$ est un plongement de lamination.

Théorème 0.2. *Soit (A, Σ) un espace stratifié supportant une structure de treillis $\mathcal{T}$. Soit i un plongement $\mathcal{T}$-controlé de (A, Σ) dans une variété M. On identifie A, Σ et $\mathcal{T}$ avec leur image dans M. Soit f un C^1-endomorphisme de M préservant Σ et vérifiant pour chaque strate $X \in \Sigma$:*

(i) *f dilate normalement X et est expansif par plaques sur X,*

il existe un voisinage V_X de X dans $\mathcal{L}_X$ vérifiant que

(ii) *chaque plaque de $\mathcal{L}_X$ incluse dans V_X est envoyée par f dans une feuille de $\mathcal{L}_X$,*
(iii) *pour chaque strate $X \in \Sigma$, il existe $\epsilon > 0$ et un voisinage V_X de X, tels que toute ϵ-pseudo-orbite de V_X qui respecte $\mathcal{L}_X$ est contenue dans X.*

Alors, pour f' C^1-proche de f, il existe un plongement $\mathcal{T}$-contrôlé i' proche de l'inclusion canonique tel que f' vérifie les propriétés (i), (iii) *et* (iii) *énumérées ci-dessus, pour l'identification de (A, Σ) via le plongement i'. De plus, la stratification de laminations Σ est persistante.*

Ce théorème a pour corollaire la persistance, en tant que stratifications, des sous-variétés à bord ou à coins normalement dilatés, ainsi que la persistance de beaucoup de stratifications de laminations présentes en dynamique produit et même la persistance de fibrés « normalement axiome A ».

1. Normally expanded laminations

Recall that a (d-dimensional)-lamination is a second countable metric space locally modeled (via compatible charts) on the product of $\mathbb{R}^d$ with a locally compact space (see [1], p. 32).

An *embedding* of a lamination $\mathcal{L}$ into a manifold M is a homeomorphism i onto its image which is leafwise differentiable with continuous and injective derivative on $\mathcal{L}$. Let f be a C^1*-endomorphism of Riemannian manifold M*:

f is continuously differentiable, non-necessarily bijective and may have critical points. We suppose that f *preserves* an embedded lamination $\mathcal{L}$: f sends each leaf of $\mathcal{L}$ into a leaf of $\mathcal{L}$. Let $T\mathcal{L}$ be the subbundle of $TM_{|\mathcal{L}}$ whose fibers are the tangent spaces to the leaves of $\mathcal{L}$.

We say that f is *normally expanding* to $\mathcal{L}$ if there exist $\lambda > 1$ and a continuous positive function C on $\mathcal{L}$ such that, for any $x \in \mathcal{L}$, any unitary vectors $v_0 \in T_x\mathcal{L}$ and $v_1 \in (T_x\mathcal{L})^{\perp}$, any $n \geqslant 0$, we have

$$\left\| p \circ Tf^n(v_1) \right\| \geqslant C(x) \cdot \lambda^n \cdot \left(1 + \left\| Tf^n(v_0) \right\|\right), \quad \text{with } p \text{ the orthogonal projection of } TM_{|\mathcal{L}} \text{ onto } T\mathcal{L}^{\perp}.$$

This embedded lamination is said to be *persistent* if, for every endomorphism f' C^1-close to f, there exists an embedding i' of $\mathcal{L}$ into M close to i, such that f' preserves the lamination $\mathcal{L}$ embedded by i' and, moreover, the dynamics induced by f and f' in the space of the leaves of $\mathcal{L}$ are the same.

In the diffeomorphism context, Hirsch, Pugh and Shub [3] have shown that normally hyperbolic and plaque expansive[1] laminations are persistent. Our first result is:

Theorem 1.1. *Let $\mathcal{L}$ be a lamination embedded in a Riemannian manifold M. Let f be a C^1-endomorphism of M which is normally expanding to $\mathcal{L}$ and plaque-expansive. Let $\mathcal{L}'$ be a precompact open subset of $\mathcal{L}$ such that the closure of $\mathcal{L}'$ is sent by f into $\mathcal{L}'$. Then $\mathcal{L}'$, endowed with the lamination structure induced by $\mathcal{L}$, is persistent.*

In particular, this theorem implies the persistence of compact normally expanded and plaque-expansive laminations. The demonstration follows a graph transform method in the spirit of [3], but with a different implementations.

2. Stratifications of normally expanded laminations

Following J. Mather [4], a *stratified space* is a compact metric space A equipped with a finite partition Σ of A into locally closed subsets, satisfying the axiom of the frontier:

$$\forall (X, Y) \in \Sigma^2, \quad \mathrm{cl}(X) \cap Y \neq \emptyset \Rightarrow \mathrm{cl}(X) \supset Y. \quad \text{We note then } X \geqslant Y.$$

The pair (A, Σ) is called *stratified space* with *support A* and *stratification Σ*.

In the same way as H. Whitney, R. Thom or J. Mather, we add a geometric structure on each stratum. A lamination structure is given on each stratum, such that for any strata $X \geqslant Y$, the dimension of the leaves of X is at least equal to the dimension of those of Y. This kind of stratified space is called *laminar* and Σ is a *stratification of laminations*. An *embedding* of this space into a manifold M is a homeomorphism i from A onto its image such that, the restriction to each stratum X is an embedding (of lamination) from X into M. We often identify the stratified space (A, Σ) with its image by the embedding i.

Example 1. Every Whitney stratification (see [6]) is a stratification of laminations.

Given a manifold M and a stratification of laminations Σ on $A \subset M$, a C^1-endomorphism f of M *preserves* Σ if f preserves each stratum $X \in \Sigma$ (endowed with its lamination structure).

Example 2. Given a C^1-diffeomorphism f of a compact manifold that satisfies Axiom A and the strong transversality condition (see [5]), if we denote by $(\Lambda_i)_i$ the spectral decomposition and $X_i := W^s(\Lambda_i)$ the laminaton whose leaves are the stable manifolds of points of Λ_i, then the partition $(X_i)_i$ is a stratification on M of laminations normally expanded by f.

Example 3. Let f be a C^1-endomorphism of a compact connected manifold M. Let K be a compact subset, f-invariant ($f^{-1}(K) = K$) and expanded by f. We endow K with the 0-dimensional lamination structure and $X := M \setminus K$ with the lamination structure of the same dimension as M. Then (K, X) is a stratification of normally expanded laminations.

[1] An endomorphism is *plaque-expansive* if for $\epsilon > 0$ small enough, for every ϵ-distant ϵ-pseudo-orbits $(x_n)_n$ and $(y_n)_n$, such that for all $n \geqslant 0$, $f(x_n)$ and x_{n+1} (resp. $f(y_n)$ and y_{n+1}) belong to a same plaque of diameter less than ϵ, then x_0 and y_0 belong to a same small plaque. In the diffeomorphism context we must consider sequences indexed by $\mathbb{Z}$.

A stratified space (A, Σ) embedded by i and preserved by $f \in End^1(M)$ is *C^1-persistent*, if for any endomorphism f' C^1-close to f, there exists an embedding i' close to i such that f' preserves the stratification Σ embedded by i'.

The main aim of our work is to show that, under some extra hypotheses explained below, the stratifications of normally expanded laminations are persistent.

2.1. Persistence of normally expanded submanifolds with boundary or corners

Our main result implies the following theorems:

Theorem 2.1. *Let N be a compact submanifold with boundary of a manifold M. Let f be a C^1-endomorphism of M which preserves and normally expands the boundary ∂N and the interior $\mathring{N}$ of N. Then the stratification $(\mathring{N}, \partial N)$ on N is C^1-persistent.*

In other words, for any map f' C^1-close to f, there exist two submanifolds $\partial N'$ and $\mathring{N}'$ such that:

- *$\mathring{N}'$ (resp. $\partial N'$) is preserved by f', diffeomorphic and C^1-close to $\mathring{N}$ (resp. ∂N) for the compact-open topology.*
- *The couple $(N' := \mathring{N}' \cup \partial N', (\mathring{N}', \partial N'))$ is a stratification and N' is the image of N by an embedding C^0-close to the canonical inclusion of N in M.*

Remark 1. Usually, N' is not a submanifold with boundary.

Recall that a compact manifold with corners N is a differentiable manifold modeled on $\mathbb{R}^d_+$. We denote by $\partial^{0_k} N$ the set of points of N which, seen in a chart, have exactly k coordinates equal to zero. The couple $(N, \Sigma := (\partial^{0_k} N)_k)$ is a stratified space.

The following theorem generalizes the one aforementioned:

Theorem 2.2. *Let N be a compact manifold with corners embedded in a manifold M. Let f be a C^1-endomorphism of M, which preserves Σ and normally expands each stratum $\partial^{0_k} N$. Then, the stratification Σ on N is C^1-persistent.*

3. Structure of trellis of laminations and main result

We constructed a very simple example of a stratification of normally expanded laminations which is not C^1-persistent. Therefore, some new conditions are necessary to imply the C^1-persistence of stratifications of normally expanded laminations.

In order to apply our main result to a stratified space (A, Σ) on a compact subset A of a manifold, we require the existence of a *tubular neighborhood* $\mathcal{L}_X$ for each stratum $X \in \Sigma$: this is a lamination supported by an open neighborhood of X in A, such that each leaf of X is a leaf of $\mathcal{L}_X$.

A similar structure was already conjectured in a local way by H. Whitney [6] in the study of analytic varieties. It was also a key ingredient constructed in the proofs by W. de Melo [2] and by C. Robinson [5] of the structural stability of diffeomorphisms that satisfy axiom A and the strong transversality condition (for the stratification of laminations defined in Example 2).

A *trellis* (*of laminations*) on (A, Σ) is a family of tubular neighborhoods $\mathcal{T} = (\mathcal{L}_X)_{X \in \Sigma}$ such that, for all strata $X \leqslant Y$:

- each plaque of $\mathcal{L}_Y$ included in $\mathcal{L}_X$ is C^1-foliated by plaques of $\mathcal{L}_X$,
- given two close points $(x, x') \in (\mathcal{L}_X \cap \mathcal{L}_Y)^2$, there exist two plaques of $\mathcal{L}_Y$ containing respectively x and x' for which such foliations are diffeomorphic and C^1-close.

An embedding i of (A, Σ) into a manifold M is *$\mathcal{T}$-controlled* if the restriction of i to $\mathcal{L}_X$ is an embedding of laminations, for all $X \in \Sigma$.

Example 4. The stratification in Example 3 admits a trellis structure. Let $\mathcal{L}_K$ be a neighborhood of K in M endowed with the 0-dimensional lamination structure. Then $(\mathcal{L}_K, X)$ is a trellis structure on $(M, (K, X))$.

We give now a restricted version of our main theorem:

Theorem 3.1. *Let (A, Σ) be a compact stratified space supporting a trellis structure $\mathcal{T}$. Let i be an embedding $\mathcal{T}$-controlled of (A, Σ) into a manifold M. We identify A, Σ and $\mathcal{T}$ with their image in M. Let f be a C^1-endomorphism of M preserving Σ and satisfying for each stratum X:*

(i) *f normally expands X and is plaque-expansive at X,*

there exists a neighborhood V_X of X in $\mathcal{L}_X$ such that

(ii) *each plaque of $\mathcal{L}_X$ included in V_X is sent by f leaf of $\mathcal{L}_X$,*
(iii) *there exists $\epsilon > 0$, such that every ϵ-pseudo orbit[2] of V_X which respects $\mathcal{L}_X$ is included in X.*

Then for every f' C^1-close to f, there exists an embedding $\mathcal{T}$-controlled i', close to i, such that for the identification of A, Σ and $\mathcal{T}$ via i', the properties (i), (ii) *and* (iii) *hold with f'. Moreover, the dynamics induced by f' on the space of the leaves of each strata of Σ is the same as the dynamics induced by f.*
In particular, the stratification of laminations Σ is C^1-persistent.

Remark 2. We do not know if property (iii) holds if conditions (i) and (ii) are satisfied. The main difficulty to apply this theorem is to build a trellis structure that satisfies (ii).

Remark 3. This result has also a version which allows A to be non-compact and/or i to be an immersion.

Remark 4. We have also a better conclusion: for every stratum X there exists a neighborhood V'_X of X in $\mathcal{L}_X$ such that, for every f' C^1-close to f, each point $i'(x) \in i'(V'_X)$ is sent by f' into the image by i' of a small plaque of $\mathcal{L}_X$ containing $f(x)$.

3.1. Products of hyperbolic rational functions

Our main result is easy to apply to many examples of product dynamics,[3] as we will see in the proof of the following example. Let

$$f : \hat{\mathbb{C}}^n \to \hat{\mathbb{C}}^n, \quad (z_i)_i \mapsto \big(R_i(z_i)\big)_i.$$

We assume that, for each i, R_i is a hyperbolic rational function of the Riemann sphere. It follows that its Julia set K_i is an expanded compact subset. The complementary X_i of K_i in $\hat{\mathbb{C}}$ is the finite union of attraction basins of the attracting periodic orbits.

Let Σ be the stratification of laminations on $\hat{\mathbb{C}}^n$, constituted by the strata $(Y_J)_{J\subset\{1,\dots,n\}}$, where the stratum Y_J is on $\prod_{j\in J} X_j \times \prod_{j\in J^c} K_j$ and of $\mathbb{R}$-dimension twice the cardinality of J. The leaves of Y_J are under the form $\prod_{j\in J} C_j \times \prod_{j\in J^c}\{k_j\}$, with C_j a connected component of $\hat{\mathbb{C}} \setminus K_j$ and with k_j a point of K_j.

The stratification Σ is C^1-persistent.

Proof. For each stratified space $(\hat{\mathbb{C}}, (K_i, X_i))$, we define a trellis structure $(\mathcal{L}_{K_i}, X_i)$, as in Example 4. We notice that properties (i), (ii) and (iii) of the main theorem are satisfied for the endomorphism R_i of $\hat{\mathbb{C}}$. Then we use by induction $n - 1$ times the following proposition:

Proposition 3.2. *Let (A, Σ) and (A', Σ') be compact stratified spaces endowed with trellis structures $\mathcal{T}$ and $\mathcal{T}'$ respectively. Let i and i' be embeddings $\mathcal{T}$ and $\mathcal{T}'$-controlled of (A, Σ) and (A, Σ) into manifolds M and M' respectively. Let f and f' be C^1 endomorphisms of, respectively, M and M', satisfying properties* (i), (ii) *and* (iii).

[2] An ϵ-pseudo-orbit $(x_n)_n \in V_X^{\mathbb{N}}$ respects $\mathcal{L}_X$ if, for all $n \geqslant 0$, $f(x_n)$ and x_{n+1} belong to a same plaque of $\mathcal{L}_X$ of diameter less than ϵ.
[3] For instance, a similar result exists on $\mathbb{R}^n$ for products of real hyperbolic polynomials.

Then the partition $\Sigma \times \Sigma'$ on $A \times A'$, whose elements are the product of a stratum of Σ with a stratum of Σ', is a stratification of laminations which is preserved by the products dynamics (f, f') of $M \times M'$. Moreover, if (f, f') normally expands this stratification, then properties (i), (ii) *and* (iii) *are satisfied for (f, f') and $\Sigma \times \Sigma'$. In particular this last stratification is persistent for every C^1-perturbation of (f, f').* □

3.2. Bundle "normally axiom A"

A consequence of the main theorem and Remark 4 is an example of a non-trivial lamination which is persistent but not normally hyperbolic:

Theorem 3.3. *Let s be a submersion of a compact manifold M onto a compact surface S. Let $\mathcal{L}$ be the lamination structure on M whose leaves are the connected components of the fibers of s.*

Let f be a diffeomorphism of M which preserves the lamination $\mathcal{L}$. Let $f_b \in \mathrm{Diff}^1(S)$ be the dynamics induced by f on the leaves spaces of $\mathcal{L}$. Suppose that:

- *f_b satisfies the axiom A and the strong transversality condition,*
- *the $\mathcal{L}$-saturated subset generated by the non-wandering set of f in M is normally hyperbolic.*

Then $\mathcal{L}$ is C^1-persistent.

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