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C. R. Acad. Sci. Paris, Ser. I 346 (2008) 545–548

COMPTES RENDUS MATHEMATIQUE

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Algebraic Geometry

Construction of a Frobenius nonsplit Harder–Narasimhan filtration

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Received 6 February 2008; accepted after revision 11 March 2008

Available online 11 April 2008

Presented by Michel Raynaud

Abstract

We construct an example of a vector bundle V over a smooth projective variety X such that for no $n \geqslant 1$, the Harder–Narasimhan filtration of $F_X^n V$ splits, where F_X is the Frobenius morphism of X. ***To cite this article: I. Biswas et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Construction d'une filtration de Harder–Narasimhan Frobenius non-scindée. Nous construisons un exemple de fibré vectoriel V au-dessus d'une variété projective lisse X tel que, pour tout $n \geqslant 1$, la filtration de Harder–Narasimhan de $F_X^n V$, où F_X est le morphisme de Frobenius de X, est non-scindée. ***Pour citer cet article : I. Biswas et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Let X be a smooth projective variety defined over an algebraically closed field k of positive characteristic. We have the geometric Frobenius morphism

$$F_X : X \longrightarrow F_k^* X,$$

where F_k is the Frobenius homomorphism of the field k. Since k is perfect, the variety $F_k^* X$ is isomorphic to X. The geometric Frobenius morphism will be considered as the self-morphism

$$F_X : X \longrightarrow X. \tag{1}$$

For any positive integer n, let F_X^n be the n-fold iteration of the self-morphism F_X of X.

Given any vector bundle V over X, for all sufficiently large integer n, the successive quotients of the Harder–Narasimhan filtration of $(F_X^n)^* V$ are strongly semistable [3, p. 259, Theorem 2.7]. In other words, there is an integer n_0, which depends on V, such that for all positive integer i, the Harder–Narasimhan filtration of $(F_X^{i+n_0})^* V$ is the pull back, by F_X^i, of the Harder–Narasimhan filtration of $(F_X^{n_0})^* V$. Using it the following can be shown.

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.03.011

If X is a curve, then for all sufficiently large n, the Harder–Narasimhan filtration of $(F_X^n)^*V$ splits into a direct sum of strongly semistable vector bundles [1, p. 356, Proposition 2.1].

It is natural to ask whether the above splitting of the Harder–Narasimhan filtration remains valid when $\dim X > 1$. An affirmative answer would have some interesting consequences, which led us to investigate it.

Our aim here is to show that the above mentioned splitting of the Harder–Narasimhan filtration does not generalize to higher dimensions. An example demonstrating this is constructed in Section 3.

2. Construction of a vector bundle

Let k be an algebraically closed field, of positive characteristic, which is uncountable. Let p be the characteristic of k. Let C be an irreducible smooth projective curve defined over k, with $\text{genus}(C) \geqslant 2$, such that the homomorphism

$$F_C^* : H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, \mathcal{O}_C) \tag{2}$$

is an isomorphism, where $F_C : C \longrightarrow C$ is the Frobenius morphism (as in (1)).

The above condition means that C is an ordinary curve (see [2, p. 208, Définition (4.12)]).

Let $\mathcal{M}_C(2)$ denote the moduli space of semistable vector bundles W of rank two and degree zero. It is an irreducible normal projective variety of dimension $4 \cdot \text{genus}(C) - 3$.

For each $n \geqslant 1$, let

$$0 \longrightarrow \mathcal{O}_C \longrightarrow (F_C^n)_* \mathcal{O}_C \longrightarrow B_n \longrightarrow 0 \tag{3}$$

be the short exact sequence of vector bundles, where the injective homomorphism is induced by the identification $(F_C^n)^* \mathcal{O}_C = \mathcal{O}_C$.

Since the homomorphism in (2) is an isomorphism, the composition

$$(F_C^n)^* = \overbrace{F_C^* \circ \cdots \circ F_C^*}^{n\text{-times}} : H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, \mathcal{O}_C) \tag{4}$$

is also an isomorphism. Using the natural identification

$$H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^1\big(C, (F_C^n)_* \mathcal{O}_C\big),$$

which is ensured by the finiteness of the morphism F_C^n, the isomorphism $(F_C^n)^*$ in (4) gives an isomorphism

$$\psi_n : H^1(C, \mathcal{O}_C) \longrightarrow H^1\big(C, (F_C^n)_* \mathcal{O}_C\big). \tag{5}$$

The homomorphism ψ_n in (5) coincides with the one associated to the inclusion homomorphism in (3). Since ψ_n is an isomorphism, from the long exact sequence of cohomologies associated to (3) we conclude that

$$H^1(C, B_n) = 0$$

for all $n \geqslant 1$. Since $\dim H^0(C, B_n) - \dim H^1(C, B_n) = 0$,

$$H^0(C, B_n) = 0 = H^1(C, B_n) \tag{6}$$

for all $n \geqslant 1$.

Using the semicontinuity of $\dim H^i$ together with (6) it follows that there is a Zariski open dense subset

$$U_n \subset \mathcal{M}_C(2) \tag{7}$$

containing the trivial vector bundle $\mathcal{O}_C \bigoplus \mathcal{O}_C$ such that for all $E \in U_n$,

$$H^0(C, B_n \otimes E) = 0 = H^1(C, B_n \otimes E). \tag{8}$$

Set

$$\mathcal{S} := \bigcap_{n=1}^{\infty} U_n, \tag{9}$$

where U_n are as in (7). Since k is uncountable, $\mathcal{S}$ is nonempty.

Take any $E \in \mathcal{S}$. Let

$$0 \longrightarrow E \longrightarrow \left(F_C^n\right)_* \left(F_C^n\right)^* E = E \otimes \left(F_C^n\right)_* \left(F_C^n\right)^* \mathcal{O}_C \longrightarrow B_n \otimes E \longrightarrow 0 \tag{10}$$

be the short exact sequence obtained by tensoring (3) with E (the equality in (10) is given by the projection formula). Let

$$\beta_n^E : H^1(C, E) \longrightarrow H^1\left(C, \left(F_C^n\right)_* \left(F_C^n\right)^* E\right) = H^1\left(C, \left(F_C^n\right)^* E\right) \tag{11}$$

be the homomorphism associated to (10) (the above equality is due to the finiteness of F_C^n). Using (8) we have:

Lemma 2.1. *For each $n \geqslant 1$, the homomorphism β_n^E in* (11) *is an isomorphism.*

Let L_1 and L_2 be two line bundles over C of degree zero such that for all $m, n \in \mathbb{Z}$,

$$\left\{L_1^{\otimes m} \otimes L_2^{\otimes n} = \mathcal{O}_C\right\} \Longrightarrow \{m = 0 = n\}. \tag{12}$$

Since k is uncountable and $\operatorname{genus}(C) \geqslant 2$, such a pair of line bundles exist.

Take any $n \geqslant 1$. For any vector bundle W over C we have an injective homomorphism $(F_C^n)^* W \hookrightarrow \operatorname{Sym}^{p^n}(W)$ of vector bundles. Also, $(F_C^n)^* L = L^{\otimes p^n}$ for any line bundle L over C. Consequently, we have a short exact sequence of vector bundles over C

$$0 \longrightarrow \left(F_C^n\right)^* (L_1 \oplus L_2) \longrightarrow \operatorname{Sym}^{p^n}(L_1 \oplus L_2) \longrightarrow \bigoplus_{i=1}^{p^n-1} L_1^{\otimes i} \otimes L_2^{\otimes (p^n - i)} \longrightarrow 0. \tag{13}$$

From (12) we have

$$H^0\left(C, \bigoplus_{i=1}^{p^n-1} L_1^{\otimes i} \otimes L_2^{\otimes (p^n - i)}\right) = 0.$$

Hence the homomorphism

$$H^1\left(C, \left(F_C^n\right)^* (L_1 \oplus L_2)\right) \longrightarrow H^1\left(C, \operatorname{Sym}^{p^n}(L_1 \oplus L_2)\right) \tag{14}$$

associated to (13) is injective.

Let

$$U_n' \subset \mathcal{M}_C(2) \tag{15}$$

be the Zariski open dense subset containing $L_1 \bigoplus L_2$ such that for all $E \in U_n'$, the homomorphism

$$\gamma_n^E : H^1\left(C, \left(F_C^n\right)^* E\right) \longrightarrow H^1\left(C, \operatorname{Sym}^{p^n}(E)\right) \tag{16}$$

associated to the natural inclusion $(F_C^n)^* E \hookrightarrow \operatorname{Sym}^{p^n}(E)$ is injective (the openness of U_n' follows from semicontinuity).

Now define

$$\mathcal{S}_0 := \left(\bigcap_{n=1}^{\infty} U_n'\right) \cap \mathcal{S} = \bigcap_{n=1}^{\infty} (U_n \cap U_n') \subset \mathcal{M}_C(2), \tag{17}$$

where U_n, U_n' and $\mathcal{S}$ are constructed in (7), (15) and (9) respectively. Since k is uncountable, the countable intersection $\mathcal{S}_0$ of open dense subsets is nonempty.

Lemma 2.2. *Take any $E \in \mathcal{S}_0$. Then the homomorphism γ_n^E in* (16) *is injective for all $n \geqslant 1$. Also, Lemma* 2.1 *holds for E.*

3. An example

We continue with the notation of Section 2.

Take any $E \in \mathcal{S}_0$ constructed in (17). Let

$$X := \mathbb{P}(E) \xrightarrow{\phi} C \tag{18}$$

be the corresponding projective bundle. Fix any very ample line bundle ξ on X. All semistable vector bundles on X considered here will be with respect to ξ.

Since $\phi_*\mathcal{O}_{\mathbb{P}(E)}(1) = E$ and $R^1\phi_*\mathcal{O}_{\mathbb{P}(E)}(1) = 0$, we have

$$H^1\big(X, \mathcal{O}_X(1)\big) = H^1(C, E) \neq 0.$$

Fix

$$\theta \in H^1\big(X, \mathcal{O}_{\mathbb{P}(E)}(1)\big) \setminus \{0\}. \tag{19}$$

Let V be the vector bundle over X that fits in the exact sequence

$$0 \longrightarrow \mathcal{O}_X(1) \longrightarrow V \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{20}$$

for which the extension class is θ. Since degree$(E) = 0$, it follows that degree$(\mathcal{O}_X(1)) := c_1(\mathcal{O}_X(1)) \cdot c_1(\xi) > 0$. Hence the vector bundle V in (20) is not semistable, and the sub-bundle $\mathcal{O}_X(1)$ in (20) gives the Harder–Narasimhan filtration of V. More precisely, for all $n \geqslant 1$, the pull back

$$\big(F_X^n\big)^*\mathcal{O}_X(1) \subset \big(F_X^n\big)^*V$$

of (20) is the Harder–Narasimhan filtration of $(F_X^n)^*V$.

We note that $(F_X^n)^*\mathcal{O}_X(1) = \mathcal{O}_X(p^n)$, and $H^1(X, \mathcal{O}_X(p^n)) = H^1(C, \mathrm{Sym}^{p^n}(E))$. The homomorphism

$$\big(F_X^n\big)^* : H^1\big(X, \mathcal{O}_X(1)\big) \longrightarrow H^1\big(X, \mathcal{O}_X(p^n)\big)$$

coincides the composition

$$\gamma_n^E \circ \beta_n^E : H^1(C, E) \longrightarrow H^1\big(C, \mathrm{Sym}^{p^n}(E)\big)$$

(see (11) and (16)). But we have already proved that $\gamma_n^E \circ \beta_n^E$ is injective (see Lemma 2.2). In particular,

$$\big(F_X^n\big)^*(\theta) \neq 0$$

for all $n \geqslant 1$, where θ is the cohomology class in (19). Hence for each $n \geqslant 1$, the pull back of the exact sequence in (20)

$$0 \longrightarrow \big(F_X^n\big)^*\mathcal{O}_X(1) \longrightarrow \big(F_X^n\big)^*V \longrightarrow \big(F_X^n\big)^*\mathcal{O}_X = \mathcal{O}_X \longrightarrow 0 \tag{21}$$

does not split.

We note that any $\mathcal{O}_X$-linear homomorphism between two vector bundles over X which is an isomorphism outside some finitely many points of X is actually an isomorphism over X between vector bundles. Therefore, for each $n \geqslant 1$, the exact sequence in (21) does not split even outside some finitely many points.

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