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Homological Algebra

Hopf type formulas for cyclic homology

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Abstract

We fit the cyclic homology of associative algebras into the context of cotriple homology of Barr and Beck. Consequently, we describe the cyclic homology of associative algebras in terms of the generalised Hopf type formulas. This Note is part of a joint project with Donadze about derived functors in cyclic (co)homology. ***To cite this article: N. Inassaridze, M. Ladra, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Formules de type Hopf pour l'homologie cyclique. On inscrit l'homologie cyclique des algèbres associatives dans le cadre de l'homologie cotriple de Barr et Beck. En conséquence, on décrit l'homologie cyclique des algèbres associatives au moyen des formules de Hopf généralisées. Cette Note fait partie d'un projet commun avec Donadze sur les foncteurs dérivés en (co)homologie cyclique. ***Pour citer cet article : N. Inassaridze, M. Ladra, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Il est bien connu que d'une part la K-théorie algébrique [12,9] et, d'autre part, la (co)homologie de Eilenberg–MacLane, la (co)homologie de Hochschild et de Shukla des algèbres associatives, la (co)homologie de André–Quillen des algèbres commutatives et la (co)homologie de Cartan–Eilenberg et de Chevalley–Eilenberg des algèbres de Lie [1–3] ont été interprétées sous forme de foncteurs dérivés non-abéliens (cf. [7]). Le but de cette Note est de décrire une philosophie analogue pour l'homologie cyclique des algèbres associatives. Une application de ce résultat produit des formules de type Hopf dans le sens de Brown et Ellis [4], en utilisant la méthode des n-multiples foncteurs dérivés de Čech, développée dans [5].

Soit k un anneau commutatif fixe, contenant le corps $\mathbb{Q}$ des nombres rationnels, Mod la catégorie des k-modules et $\otimes = \otimes_k$. Les algèbres sont des k-algèbres (non-unitaires) associatives et on note Alg leur catégorie. Par '*algèbre libre*' on entend une algèbre libre (non-unitaire) sur un certain module. Étant données une algèbre A et ses sous-algèbres

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B et C, on note $[B, C]$ le module des commutateurs additifs et $\mathcal{A}b$: Alg $\to$ Mod le foncteur additif d'abélianisation défini par $\mathcal{A}b(A) = A/[A, A]$.

Considérons le cotriple 'libre' $\mathcal{F} = (F, \tau, \delta)$ dans la catégorie Alg. Pour un objet $A \in$ Alg la *$\mathcal{F}$-cotriple résolution* de A est un objet simplicial augmenté dans la catégorie Alg, où $F_n(A) = F^{n+1}(A) = F(F^n(A))$, $d_i^n = F^i(\tau_{F^{n-i}})$ et $s_i^n = F^i(\delta_{F^{n-i}})$, $0 \leqslant i \leqslant n$.

Soit T : Alg $\to$ Mod un foncteur covariant. Le n-ième foncteur dérivé à gauche non-abélien $L_n^{\mathcal{F}} T$: Alg $\to$ Mod, $n \geqslant 0$ (par rapport au cotriple $\mathcal{F}$) est donné par

$$L_n^{\mathcal{F}} T(A) = H_n\big(T\big(F_*(A)\big)\big) \quad \text{et} \quad L_n^{\mathcal{F}} T(\alpha) = H_n\big(T\big(F_*(\alpha)\big)\big)$$

pour tout objet $A \in$ Alg et tout morphisme $\alpha \in$ Alg.

Proposition 1.1. *Soit A une algèbre, projective comme module. Alors la n-ième homologie cyclique de l'algèbre A est isomorphe à la valeur du n-ième foncteur cotriple dérivé du foncteur additif d'abélianisation pour A, i.e.*

$$HC_n(A) \cong L_n^{\mathcal{F}} \mathcal{A}b(A), \quad n \geqslant 0.$$

Étant donnée une algèbre A et $m \geqslant 1$, l'algèbre linéaire spéciale de Lie $sl_m(A)$ est l'algèbre des $m \times m$ matrices à valeurs dans A de trace zéro (la trace est évaluée dans $A/[A, A]$).

Corollaire 1.2. *Soit A une algèbre, projective comme module. Alors on a un isomorphisme*

$$HC_{n+1}(A) \cong L_n^{\mathcal{F}} sl_m(A), \quad m \geqslant 1,\ n \geqslant 1.$$

Comme application de la Proposition 1.1 on obtient les formules généralisées de type Hopf pour l'homologie cyclique des algèbres, en utilisant la méthode des n-multiples foncteurs dérivés de Čech [5,8].

Étant donné un entier n non-négatif, on note $\mathcal{C}_n$ la catégorie définie par l'ensemble ordonné de tous les sous-ensembles de l'ensemble $\langle n\rangle = \{1, \ldots, n\}$. Un n-cube d'algèbres est un foncteur $\mathfrak{F} : \mathcal{C}_n \to$ Alg. Étant donné un n-cube d'algèbres $\mathfrak{F}$, on voit aisément qu'il existe un homomorphisme naturel $\mathfrak{F}_X \xrightarrow{\alpha_X} \lim_{Y \supset X} \mathfrak{F}_Y$ pour chaque $X \subseteq \langle n\rangle$, $X \neq \langle n\rangle$.

Soit A une algèbre. Un n-cube d'algèbres $\mathfrak{F}$ est appelé une *n-présentation* de l'algèbre A si $\mathfrak{F}_{\langle n\rangle} = A$. Une n-présentation $\mathfrak{F}$ de A est appelée *libre* si l'algèbre $\mathfrak{F}_X$ est libre pour chaque $X \neq \langle n\rangle$ et elle est appelée *exacte* si l'homomorphisme α_X a une section k-linéaire pour chaque $X \neq \langle n\rangle$.

Notre résultat principal a la forme suivante :

Théorème 2.2. *Soit A une algèbre, projective comme module, et $\mathcal{F}$ une n-présentation libre et exacte de A. Alors on a un isomorphisme*

$$HC_n(A) \cong \Big(\bigcap_{i \in \langle n\rangle} L_i \cap [Q, Q]\Big) \Big/ \Big(\sum_{X \subseteq \langle n\rangle} \Big[\bigcap_{i \in X} L_i, \bigcap_{i \notin X} L_i\Big]\Big), \quad n \geqslant 1,$$

où $Q \trianglerighteq L_1, \ldots, L_n$ est la $(n+1)$-suite normale d'algèbres induite par $\mathfrak{F}$ de la manière suivante : $Q = \mathfrak{F}_\emptyset$ et $L_i = \ker \alpha_{\{i\}}^{\emptyset}$, $i \in \langle n\rangle$.

0. Introduction

It is well-known that algebraic K-theory [12,9], on one hand, and Eilenberg–MacLane group (co)homology, Hochschild and Shukla (co)homology of associative algebras, André–Quillen (co)homology of commutative algebras, Cartan–Eilenberg and Chevalley–Eilenberg (co)homology of Lie algebras [1–3], on the other hand, have been described in terms of non-Abelian derived functors (see also [7]). The purpose of this Note is to describe an analogous philosophy for the cyclic homology of associative algebras. As an application of this result we obtain the Hopf type formulas for the cyclic homology in the sense of Brown and Ellis [4], using the method of n-fold Čech derived functors developed in [5].

Details of the results given here will appear in a paper in preparation.

Let k denote a fixed commutative ring containing the field $\mathbb{Q}$ of rational numbers, Mod the category of k-modules and $\otimes = \otimes_k$. Algebras are (non-unital) associative k-algebras and their category is denoted by Alg. The term '*free algebra*' means a free (non-unital) algebra over some module. Given an algebra A and its subalgebras B and C, let $[B, C]$ denote the module of additive commutators and $\mathcal{A}b : \mathrm{Alg} \to \mathrm{Mod}$ the additive Abelianization functor given by $\mathcal{A}b(A) = A/[A, A]$.

1. Cyclic homology via derived functors

Let us consider the cotriple $\mathcal{F} = (F, \tau, \delta)$ in the category Alg constructed in the following way: let $F : \mathrm{Alg} \to \mathrm{Alg}$ be the endofunctor defined as follows: for an object A of Alg, let $F(A)$ denote the free algebra on the underlying module A; for a morphism $\alpha : A \to A'$ of Alg, let $F(\alpha)$ be the canonical algebra homomorphism from $F(A)$ to $F(A')$ induced by α. Let $\tau : F \to 1_{\mathrm{Alg}}$ be the obvious natural transformation and let $\delta : F \to F^2$ be the natural transformation induced for every $A \in \mathrm{Alg}$ by the natural inclusion of modules $A \hookrightarrow F(A)$. Given an object $A \in \mathrm{Alg}$, the $\mathcal{F}$-*cotriple resolution* of A is an augmented simplicial object, $(F_*(A), d_0^0, A)$, in the category Alg, where $F_n(A) = F^{n+1}(A) = F(F^n(A))$, $d_i^n = F^i(\tau_{F^{n-i}})$, $s_i^n = F^i(\delta_{F^{n-i}})$, $0 \leqslant i \leqslant n$.

Lemma 1.1. *Let A be an algebra.*

(i) *$(F_*(A), d_0^0, A)$ is acyclic.*
(ii) *If A is a projective k-module, then $(F_*(A)^{\otimes n}, {d_0^0}^{\otimes n}, A^{\otimes n})$, $n \geqslant 1$, is acyclic.*

Let $T : \mathrm{Alg} \to \mathrm{Mod}$ be a covariant functor. The n-th *non-Abelian left derived functor* $L_n^{\mathcal{F}} T : \mathrm{Alg} \to \mathrm{Mod}$, $n \geqslant 0$ (relative to the cotriple $\mathcal{F}$) is given by

$$L_n^{\mathcal{F}} T(A) = H_n\big(T\big(F_*(A)\big)\big) \quad \text{and} \quad L_n^{\mathcal{F}} T(\alpha) = H_n\big(T\big(F_*(\alpha)\big)\big)$$

for any object $A \in \mathrm{Alg}$ and any morphism $\alpha \in \mathrm{Alg}$.

Let us recall that the cyclic homology $HC_*(A)$ of an algebra A is the homology of the following Connes' complex $C_*^{\lambda}(A)$:

$$\cdots \longrightarrow A^{\otimes(n+1)}/(1-t) \xrightarrow{b} A^{\otimes n}/(1-t) \xrightarrow{b} \cdots \xrightarrow{b} A^{\otimes 2}/(1-t) \xrightarrow{b} A,$$

where the Hochschild boundary map b is given by the formula

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n (a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})$$

and $t : A^{\otimes(n+1)} \to A^{\otimes(n+1)}$, $n \geqslant 0$, is the cyclic operator given by

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n (a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}).$$

Proposition 1.1. *Let A be an algebra, which is projective as a module. Then the n-th cyclic homology of the algebra A is isomorphic to the value of the n-th cotriple derived functor of the additive Abelianization functor on A, i.e.*

$$HC_n(A) \cong L_n^{\mathcal{F}} \mathcal{A}b(A), \quad n \geqslant 0.$$

Proof. Consider the following bicomplex $\mathbb{L}$ of modules

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle b} & & \\
C_2^\lambda(F_0(A)) & \longleftarrow & C_2^\lambda(F_1(A)) & \longleftarrow & C_2^\lambda(F_2(A)) & \longleftarrow & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle b} & & \\
C_1^\lambda(F_0(A)) & \longleftarrow & C_1^\lambda(F_1(A)) & \longleftarrow & C_1^\lambda(F_2(A)) & \longleftarrow & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle b} & & \\
C_0^\lambda(F_0(A)) & \longleftarrow & C_0^\lambda(F_1(A)) & \longleftarrow & C_0^\lambda(F_2(A)) & \longleftarrow & \cdots,
\end{array} \tag{1}$$

obtained by applying Connes' complex to the $\mathcal{F}$-cotriple resolution $(F_*(A), d_0^0, A)$ of the algebra A with appropriate sign. Using Lemma 1.1, from (1) we deduce that $H_q^v H_0^h(\mathbb{L}) = HC_q(A)$ and $H_q^v H_p^h(\mathbb{L}) = 0$ for any $q \geqslant 0$, $p > 0$. On the other hand, by [10, Proposition 5.4] we have the equalities $H_p^h H_0^v(\mathbb{L}) = L_p^{\mathcal{F}} \mathcal{A}b(A)$ and $H_p^h H_q^v(\mathbb{L}) = 0$ for any $p \geqslant 0$, $q > 0$. Now the bicomplex spectral sequence argument completes the proof. $\square$

Given an algebra A and $m \geqslant 1$, the general linear Lie algebra $gl_m(A)$ is the Lie algebra of $m \times m$ matrices with entries in A, while the special linear Lie algebra $sl_m(A)$ is the Lie algebra of $m \times m$ matrices of trace zero (the trace being evaluated in $A/[A, A]$). It is clear that $sl_m(A)$ is an ideal in $gl_m(A)$ and there is the following short exact sequence of Lie algebras

$$0 \longrightarrow sl_m(A) \longrightarrow gl_m(A) \xrightarrow{\text{tr}} A/[A, A] \longrightarrow 0. \tag{2}$$

Thanks to (2) and the fact that the augmented simplicial module $(gl_m(F_*(A)), gl_m(d_0^0), gl_m(A))$ is acyclic for the $\mathcal{F}$-cotriple resolution $(F_*(A), d_0^0, A)$ of the algebra A, we have

Corollary 1.2. *Let A be an algebra, which is projective as a module. Then there is an isomorphism*

$$HC_{n+1}(A) \cong L_n^{\mathcal{F}} sl_m(A), \quad m \geqslant 1,\ n \geqslant 1.$$

2. Hopf type formulas

An application of Proposition 1.1 is to obtain the generalised *Hopf type formulas* for the cyclic homology of algebras, using the method of n-fold Čech derived functors [5,8]. We can think as an initial result in this direction the exact sequence

$$HC_1(Q) \longrightarrow HC_1(A) \longrightarrow L/[L, Q] \longrightarrow Q/[Q, Q] \longrightarrow A/[A, A] \longrightarrow 0 \tag{3}$$

(see [11]), where $Q \xrightarrow{\alpha} A$ is a surjective homomorphism of algebras and $L = \operatorname{Ker}\alpha$. In case α is a free presentation of the algebra A, then (3) induces the Hopf formula for the first cyclic homology

$$HC_1(G) \cong \big(L \cap [Q, Q]\big)/[Q, L]. \tag{4}$$

We generalise the formula (4) to any dimension.

Given a non-negative integer n, we denote by $\mathcal{C}_n$ the category determined by the ordered set of all subsets of the set $\langle n \rangle = \{1, \ldots, n\}$. An *n-cube of algebras* is a functor $\mathfrak{F} : \mathcal{C}_n \to \text{Alg}$, and we will denote its component parts by $X \mapsto \mathfrak{F}_X$, $\rho_Y^X \mapsto \alpha_Y^X$.

Example 1. Let (A_*, d_0^0, A) be an augmented simplicial algebra. A natural n-cube of algebras $A^{(n)} : \mathcal{C}_n \to \text{Alg}$, $n \geqslant 1$ is defined in the following way:

$$A_A^{(n)} = A_{n-1-|X|} \quad \text{for all } X \subseteq \langle n \rangle \quad \text{and} \quad \alpha_{X\cup\{j\}}^X = d_{k-1}^{n-1-|X|} \quad \text{for all } X \neq \langle n \rangle,\ j \notin X,$$

where $A_{-1} = A$, $\delta(k) = j$ and $\delta : \langle n - |X| \rangle \to \langle n \rangle \setminus X$ is the unique monotone bijection.

Given an n-cube of algebras $\mathfrak{F}$, it is easy to see that there exists a natural homomorphism $\mathfrak{F}_X \xrightarrow{\alpha_X} \lim_{Y \supset X} \mathfrak{F}_Y$ for any $X \subseteq \langle n \rangle$, $X \neq \langle n \rangle$.

Let A be an algebra. An n-cube of algebras $\mathfrak{F}$ will be called an *n-presentation* of the algebra A if $\mathfrak{F}_{\langle n \rangle} = A$. An n-presentation $\mathfrak{F}$ of A is called *free* if the algebra $\mathfrak{F}_X$ is free for any $X \neq \langle n \rangle$ and called *exact* if the homomorphism α_X has a k-linear splitting for any $X \neq \langle n \rangle$.

Lemma 2.1. *Let (F_*, d_0^0, A) be an augmented (pseudo) simplicial algebra (see [7]). Then (F_*, d_0^0, A) is a free (pseudo) simplicial resolution of A if and only if the n-cube of algebras $F^{(n)}$ is a free exact n-presentation of A for any $n \geqslant 1$.*

Given an algebra A and a 1-presentation of A, i.e., a homomorphism of algebras $\alpha : Q \to A$, the Čech augmented complex $(\check{C}(\alpha)_*, \alpha, A)$ for α is given, by

$$\check{C}(\alpha)_n = \underbrace{Q \times_A \cdots \times_A Q}_{(n+1)\text{-times}} = \{(x_0, \ldots, x_n) \in Q^{n+1} | \alpha(x_0) = \cdots = \alpha(x_n)\},$$

$d_i^n(x_0, \ldots, x_n) = (x_0, \ldots, \hat{x_i}, \ldots, x_n)$ and $s_i^n(x_0, \ldots, x_n) = (x_0, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n)$, $0 \leqslant i \leqslant n$ (see [7]).

Now let $\mathfrak{F}$ be an n-presentation of the algebra A. Applying $\check{C}$ above, in the n-independent directions, this construction leads naturally to an augmented n-simplicial algebra. The diagonal of this augmented n-simplicial algebra gives the augmented simplicial algebra $(\check{C}^{(n)}(\mathfrak{F})_*, \alpha, A)$ called an *augmented n-fold Čech complex for* $\mathfrak{F}$, where $\alpha = \alpha_{\langle n \rangle}^{\emptyset} : \mathfrak{F}_\emptyset \to A$. In case $\mathfrak{F}$ is a free exact n-presentation of the algebra A, then $(\check{C}^{(n)}(\mathfrak{F})_*, \alpha, A)$ will be called an *n-fold Čech resolution of* A. Given a functor $T : \mathrm{Alg} \to \mathrm{Mod}$, define *$i$-th n-fold Čech derived functor* $\mathcal{L}_i^{n\text{-fold}} T : \mathrm{Alg} \to \mathrm{Mod}$, $i \geqslant 0$, of T by choosing for each $A \in \mathrm{Alg}$, a free exact n-presentation $\mathfrak{F}$ and setting

$$\mathcal{L}_i^{n\text{-fold}} T(A) = H_i\big(T\check{C}^{(n)}(\mathfrak{F})_*\big).$$

Note that by the comparison theorem [8] (see also [5, Theorem 16]) the n-fold Čech derived functors are well defined. An n-cube of algebras $\mathfrak{F}$ induces a normal $(n+1)$-ad of algebras $Q \trianglerighteq L_1, \ldots, L_n$, where $Q = \mathfrak{F}_\emptyset$ and $L_i = \ker \alpha_{\{i\}}^{\emptyset}$, $i \in \langle n \rangle$.

Proposition 2.1. *Let A be an algebra. Then there is an isomorphism*

$$\mathcal{L}_n^{n\text{-fold}} \mathcal{A}b(A) \cong \Big(\bigcap_{i \in \langle n \rangle} L_i \cap [Q, Q]\Big) \Big/ \Big(\sum_{X \subseteq \langle n \rangle} \Big[\bigcap_{i \in X} L_i, \bigcap_{i \notin X} L_i\Big]\Big), \quad n \geqslant 1,$$

where $Q \trianglerighteq L_1, \ldots, L_n$ is the normal $(n+1)$-ad of algebras induced by a free exact n-presentation $\mathfrak{F}$ of the algebra A.

Theorem 2.2. *Let A be an algebra, which is projective as a module and $\mathcal{F}$ a free exact n-presentation of A. Then there is an isomorphism*

$$HC_n(A) \cong \Big(\bigcap_{i \in \langle n \rangle} L_i \cap [Q, Q]\Big) \Big/ \Big(\sum_{X \subseteq \langle n \rangle} \Big[\bigcap_{i \in X} L_i, \bigcap_{i \notin X} L_i\Big]\Big), \quad n \geqslant 1,$$

where $Q \trianglerighteq L_1, \ldots, L_n$ is the normal $(n+1)$-ad of algebras induced by $\mathfrak{F}$.

Sketch of the proof. Let (F_*, d_0^0, A) be a free (pseudo)simplicial resolution of the algebra A and consider the short exact sequence of (pseudo)simplicial modules $0 \to [F_*, F_*] \to F_* \to \mathcal{A}b(F_*) \to 0$. By the induced long exact homology sequence we have the isomorphisms of modules

$$L_n^{\mathcal{F}} \mathcal{A}b(A) \cong \Big(\bigcap_{i \in \langle n \rangle} \operatorname{Ker} \overline{d_{i-1}^{n-1}}\Big) \Big/ \Big(\overline{d_n^n}\Big(\bigcap_{i \in \langle n \rangle} \operatorname{Ker} \overline{d_{i-1}^n}\Big)\Big), \quad n \geqslant 1,$$

where $\overline{d_i^n}$ is a restriction of d_i^n to $[F_n, F_n]$. Since the shift of simplicial object F_* is the contractible augmented simplicial object $(\mathrm{Dec}(F_*), d_0^1, F_0)$ (see [6]), by Lemma 2.1 the n-cube of algebras $\mathrm{Dec}(F)^{(n)}$ is a free exact n-presentation of F_0. Using Proposition 2.1 we arrive to the equality

$$\bigcap_{i \in \langle n \rangle} \operatorname{Ker} d_{i-1}^n \cap [F_n, F_n] = \sum_{X \subseteq \langle n \rangle} \Big[\bigcap_{i \in X} \operatorname{Ker} d_{i-1}^n, \bigcap_{i \notin X} \operatorname{Ker} d_{i-1}^n\Big], \quad n \geqslant 1.$$

Now it is easy to check that

$$d_n^n\Big(\bigcap_{i\in\langle n\rangle}\operatorname{Ker}\overline{d_{i-1}^n}\Big)=d_n^n\Big(\sum_{X\subseteq\langle n\rangle}\Big[\bigcap_{i\in X}\operatorname{Ker}d_{i-1}^n,\bigcap_{i\notin X}\operatorname{Ker}d_{i-1}^n\Big]\Big)=\sum_{X\subseteq\langle n\rangle}\Big[\bigcap_{i\in X}\operatorname{Ker}d_{i-1}^{n-1},\bigcap_{i\notin X}\operatorname{Ker}d_{i-1}^{n-1}\Big].$$

Therefore

$$L_n^{\mathcal{F}}\mathcal{A}b(A)\cong\Big(\bigcap_{i\in\langle n\rangle}\operatorname{Ker}d_{i-1}^{n-1}\cap[F_{n-1},F_{n-1}]\Big)\Big/\Big(\sum_{X\subseteq\langle n\rangle}\Big[\bigcap_{i\in X}\operatorname{Ker}d_{i-1}^{n-1},\bigcap_{i\notin X}\operatorname{Ker}d_{i-1}^{n-1}\Big]\Big),\quad n\geqslant 1.$$

Using Proposition 1.1, Lemma 2.1 and Proposition 2.1 complete the proof. □

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References

[1] M. Barr, Shukla cohomology and triples, J. Algebra 5 (1967) 222–231.
[2] M. Barr, Cartan–Eilenberg cohomology and triples, J. Pure Appl. Algebra 112 (1996) 219–238.
[3] M. Barr, J. Beck, Homology and standard constructions, in: B. Eckmann (Ed.), Seminar on Triples and Categorical Homology Theory, in: Lecture Notes in Math., vol. 80, Springer-Verlag, Berlin/New York, 1969, pp. 245–335.
[4] R. Brown, G.J. Ellis, Hopf formulae for the higher homology of a group, Bull. London Math. Soc. 20 (1988) 124–128.
[5] G. Donadze, N. Inassaridze, T. Porter, N-fold Čech derived functors and generalised Hopf type formulas, K-Theory 35 (2005) 341–373.
[6] J. Duskin, Simplicial methods and the interpretation of “triple” cohomology, Mem. Amer. Math. Soc. 163 (1975).
[7] H. Inassaridze, Non-Abelian Homological Algebra and its Applications, Kluwer Academic Publishers, Dordrecht, 1997.
[8] N. Inassaridze, N-fold Čech derived functors of group valued functors, Bull. Georgian Acad. Sci. 168 (2) (2003).
[9] F. Keune, Derived functors and algebraic K-theory, in: H. Bass (Ed.), Algebraic K-Theory I. Higher K-Theories, in: Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin, 1973, pp. 166–176.
[10] J.-L. Loday, D. Quillen, Cyclic homology and the Lie algebra homology of matrices, Comment. Math. Helv. 59 (4) (1984) 569–591.
[11] D. Quillen, Cyclic cohomology and algebra extensions, K-Theory 3 (1989) 205–246.
[12] R.G. Swan, Some relations between higher K-functors, J. Algebra 21 (1972) 113–136.