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Rigidity of magnetic flows for compact surfaces

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Abstract

Let $\phi_t : T_1M \to T_1M$ be the magnetic flow of the pair (g, Ω). We show that if ϕ_t preserves a $C^{2,1}$ codimension one foliation then (M, g) has constant, nonpositive Gaussian curvature and Ω is a constant multiple of the area form of (M, g). So if the genus of M is greater than one, the flow is either Anosov or conjugate to a horocycle flow. If M is a torus, the flow is actually geodesic and flat. ***To cite this article: J.B. Gomes, R.O. Ruggiero, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Rigidité des flots magnétiques sur des surfaces compactes. Soit $\phi_t : T_1M \to T_1M$ le flot magnétique du pair (g, Ω). Nous demonstrons que si ϕ_t preserve un feuilletage $C^{2,1}$ de codimension 1, alors la courbure de (M, g) est une constante non positive et la forme Ω est le produit d'une constante par la forme d'aire de (M, g). ***Pour citer cet article : J.B. Gomes, R.O. Ruggiero, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Magnetic flows are mathematical models for the motion of a particle under the action of magnetic fields. Geodesic (trivial magnetic fields) and horocycle flows are particular examples of magnetic fields. Motivated by the famous work of E. Ghys [2], who classified Anosov flows in circle bundles having C^2 foliations, we study rigidity phenomena in magnetic flows preserving a highly smooth codimension one foliation. The C^2 assumption on the weak invariant foliations of an Anosov flow can be improved to $C^{1,1}$ by the work of Hurder and Katok [6], namely, C^1 smoothness with Lipschitz derivatives. The main result of the Note is the following, where $C^{2,1}$ means C^2 with Lipschitz C^2 derivatives:

Theorem A. *Let $\phi_t : T_1M \to T_1M$ be the magnetic flow of the pair (g, Ω) restricted to the unit tangent bundle of a closed surface M. Suppose that ϕ_t preserves a $C^{2,1}$ codimension one foliation. Then (M, g) has constant, nonpositive*

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Gaussian curvature and Ω is a constant multiple of the area form of (M, g). Moreover, if the surface has genus greater than one the magnetic flow is either Anosov or conjugate to a horocycle flow. If the surface is the torus, the magnetic flow is actually the geodesic flow of a flat torus.

Theorem A extends a result by G. Paternain [8] for Anosov magnetic flows of compact surfaces of genus greater than one, with $C^{1,1}$ weak foliations; and a result by Gomes and Ruggiero [4] for geodesic flows of compact surfaces which preserve $C^{2,1}$ codimension one foliations.

2. Magnetic flows on surfaces and the Riccati equation

Let (M, g) be a C^∞ Riemannian manifold. The tangent bundle of M will be denoted by TM, the cotangent bundle of M will be denoted by T^*M, and the unit tangent bundle of (M, g) by T_1M. The map $\pi : TM \to M$ will be the canonical projection $\pi(p, v) = p$, where $p \in M$ and v is a vector tangent to M at p. For $\theta = (p, v)$, the vertical subspace $V_\theta \subset T_\theta TM$ is the kernel of $d_\theta \pi$, the horizontal subspace $H_\theta \subset T_\theta TM$ is the orthogonal complement of V_θ with respect to the Sasaki metric.

Now, let (S, g) be a compact surface, and Ω a closed 2-form on S. For a unit vector $v \in T_pS$, we will denote by iv the unique unit vector that is orthogonal to v and such that $\{v, iv\}$ has the canonical orientation T_pS. The Lorentz force associated to the form Ω, $L : TS \to TS$, is defined by $\Omega_x(v, w) = g_x(L_x(v), w)$. Let α be the canonical 1-form generating the geodesic flow of (S, g). The *magnetic flow* of the pair (g, Ω) is the Hamiltonian flow of the function $\mathbf{H} : TS \to \mathbb{R}$, $\mathbf{H}(p, v) = \frac{1}{2} g_p(v, v)$ with respect to the sympletic form $\omega = -\mathrm{d}\alpha + \pi^*\Omega$. Notice that the unit tangent bundle of (S, g) is a regular energy level of the magnetic flow that is invariant by the dynamics.

Let Ω_a be the area form of the surface (S, g), so there exists a unique C^∞ function $f : S \to \mathbb{R}$ such that $\Omega = f\Omega_a$. Let X and X_f denote respectively the generators of the geodesic flow for g and the magnetic flow for $(g, f\Omega_a)$. We have (see, for instance, [10]) $X_f(v) = (v, L_{\pi(v)}(v))$ and $X_f = X + f\mathcal{V}$, where $\mathcal{V}$ is a unit vector field tangent to the vertical bundle (chosen in a way that the orientation determined by the horizontal subspace and $\mathcal{V}$ is the canonical one).

The expression $q(\gamma) = k(\gamma) + f^2(\gamma) - \langle \nabla(\gamma), i\gamma' \rangle$ is called the *magnetic curvature*. Let $\tilde{J}$ be a magnetic Jacobi field defined in a magnetic geodesic γ, J its orthogonal projection into a canonically oriented, unit vector orthogonal to γ'. As in the geodesic case, the function $u = \frac{J'}{J}$, satisfies (see, for instance, [5,10]) the *magnetic Riccati equation* $u' + u^2 + q(\gamma) = 0$.

By the well known connection between invariant Lagrangian bundles and the Riccati equation (Theorem 2 in [9]), and the fact that a two dimensional, invariant subspace of T_1S is isotropic with respect to the form ω, we obtain:

Lemma 2.1. *Let (S, g) be a compact surface, and suppose that the magnetic flow of the pair (g, Ω) preserves a continuous, codimension one foliation $\mathcal{F}$ of T_1S by smooth leaves. Then the magnetic flow has no conjugate points and the bundle $\mathcal{E}$ of tangent spaces of the leaves are graphs of a continuous family of Riccati operators as follows. There exists a continuous function $U : T_1M \to \mathbb{R}$ such that*

(i) *The subspace $\mathcal{E}(\theta) = \mathcal{E} \cap N_\theta$ is the graph of the linear map $\bar{U}_\theta : H \cap N_\theta \to V \cap N_\theta$ given by $\bar{U}_\theta(Z) = U(\theta)Z$ for every $\theta \in T_1M$.*
(ii) *The function $u_\theta(t)$ given by $u_\theta(t) = U(\phi_t(\theta))$ is a solution of the Riccati equation $u' + u^2 + q = 0$, for every $\theta \in T_1M$.*

Here, $N_\theta = \{\xi \in T_\theta T_1M;\ \langle \mathrm{d}\pi \cdot \xi, \mathrm{d}\pi X_f(\theta) \rangle_{\pi(\theta)} = 0\}$ for $\theta = (p, v)$. We observe that $\mathrm{d}_\theta \pi X_f(\theta) = v$.

3. The Godbillon–Vey number and the magnetic curvature

The *Godbillon–Vey* class of a transversally oriented, codimension one foliation $\mathcal{F}$ of a three manifold M is a de Rham cohomology class $GV(\mathcal{F}) = [\eta \wedge \mathrm{d}\eta] \in \mathbb{H}^3(M, \mathbb{R})$ (for details see for instance [11]). If S is a closed oriented surface and $M = T_1S$, the real number $gv(\mathcal{F}) = \int_{T_1S} \eta \wedge \mathrm{d}\eta$ is called the *Godbillon–Vey number* of $\mathcal{F}$.

Now, let $\mathcal{F}$ be a codimension one, C^2 foliation of $T_1 S$ that is invariant by the magnetic flow. By Lemma 2.1, the tangent bundle of the foliation $\mathcal{F}$ gives us an invariant Lagrangian subbundle. So it has an attached Riccati operator $U : T_1 S \to \mathbb{R}$ as described in Lemma 2.1.

Theorem 3.1. (1) *The Godbillon–Vey number of $\mathcal{F}$ can be written as*

$$gv(\mathcal{F}) = 4\pi^2 \chi(S) - 3 \int_{T_1 S} (\mathcal{V}U)^2 \nu \wedge \alpha \wedge \beta,$$

where $\chi(S)$ is the Euler characteristic of S;

(2) *the function U is constant along the fibres, that is, $\mathcal{V}U \equiv 0$ if, and only if, the magnetic curvature q is a constant $\leqslant 0$.*

The proof of item (1) follows from Cartan's structural equations and Lemma 2.1. The proof of item (2) is made in two steps. First, we assume that $\mathcal{V}U = 0$ in $T_1 S$, namely, the Riccati operator is constant along vertical fibres. In this case the Riccati operator defines a function in the surface S, namely, $h(p) = U(\pi^{-1}(p))$, where π is the canonical projection. We would like to show that U is constant. This is a consequence of the next result and the maximum principle:

Lemma 3.2. *If $\mathcal{V}U = 0$ then the function $h : S \to \mathbb{R}$ given by $h(p) = U(\pi^{-1}(p))$ is a C^2 harmonic function.*

The second step to show item (2) of Theorem 3.1 is easier than step one. Indeed, if the magnetic curvature is constant, it is easy to show that it has to be nonpositive. If it is negative $q = -c^2$ the flow is Anosov and the Riccati operator of the foliation is either c or $-c$. If it is 0 the Riccati operator is 0 as well.

Corollary 3.3. *The Godbillon–Vey number of $\mathcal{F}$ is maximal if and only if the magnetic curvature is constant.*

4. Rigidity: the proof of the main theorem

First of all, we need two topological lemmas which are consequences of the existence of a foliation of $T_1 S$ by graphs of the canonical projection:

Lemma 4.1. *Let (S, g) be a compact surface, and suppose that there exists a C^1 codimension one foliation $\mathcal{F}$ of $T_1 S$ which is invariant under the magnetic flow of a pair (g, Ω). Then the genus of the surface is greater than 0.*

Lemma 4.2. *Let (S, g) be a compact surface of genus greater than one. Let $\mathcal{F}$ be a C^1 codimension 1 foliation of $T_1 S$ which is invariant under the magnetic flow of a pair (g, Ω). Then $\mathcal{F}$ has no compact leaves.*

Now, we can use the following result due to E. Ghys ([3], Theorem 5.3):

Theorem 4.3. *Let $\mathcal{F}$ be a $C^{2,1}$ codimension 1 foliation without compact leaves of the unit tangent bundle of a compact surface S of genus greater than 1. Then $\mathcal{F}$ is $C^{2,1}$ conjugate with a central foliation of the geodesic flow of a metric with constant curvature -1 in S.*

By this theorem and Theorem 3.8 in [6], the Godbillon–Vey number of $\mathcal{F}$ equals the Godbillon–Vey number of a hyperbolic central foliation, so we can apply item (2) of Lemma 3.1 to get,

Theorem 4.4. *Let (S, g) be a closed surface of genus greater than one. Assume that there exists a codimension one foliation $\mathcal{F}$ of class $C^{2,1}$ in $T_1 S$, whose leaves are invariant under the magnetic flow of a pair (g, Ω). Then, the magnetic curvature is a constant $\leqslant 0$.*

Next, we look at the consequences of Theorem 4.4 in the geometry of (S, g) and the form $\Omega = f \Omega_a$ defining the magnetic flow. The following result is straightforward from the definitions:

Lemma 4.5. *Suppose that the magnetic curvature $q(x, v)$ is constant. Then, the function f and the Gaussian curvature k of (S, g) are constants.*

Corollary 4.6. *Let (S, g) be a closed surface of genus greater than one. Assume that there exists a codimension one foliation $\mathcal{F}$ of class $C^{2,1}$ in T_1S, whose leaves are invariant under the magnetic flow of S. Then, either the magnetic flow is of Anosov type or is conjugate to a horocycle flow.*

Proof. By Theorem 4.4, the magnetic curvature is constant $q \leqslant 0$. If $q < 0$, the proof of Lemma 4.5 implies that $f^2 < -k$, and by results of Grognet in [5], we conclude that the flow is of Anosov type. If $q = 0$, then $k = -f^2$ and the magnetic flow is conjugate to a horocycle flow (see for instance [7]). □

Combining Theorem 4.4 and Lemma 4.5 we get Theorem A in the case where the genus of the surface is greater than one. Now, we obtain Theorem A in the torus case.

Theorem 4.7. *Let (T^2, g) be a 2-torus with a Riemannian metric g. Assume that there exists a codimension one foliation $\mathcal{F}$ of class C^1 in T_1T^2, whose leaves are invariant under the magnetic flow of a pair (g, Ω). Then $f = 0$, that is, the magnetic flow reduces to geodesic flow. Moreover, (T^2, g) has no conjugate points and therefore is flat.*

Bialy in [1] shows that for any non-zero magnetic field on a n-torus there are always orbits of the magnetic flow which have conjugate points, provided that the Riemannian metric is conformally flat. Since the uniformisation theory for surfaces yields that every metric in T^2 is conformally flat, we get, using Lemma 2.1, that the magnetic flow is in fact geodesic. Since by Lemma 2.1, the magnetic flow in the torus has no conjugate points, Hopf's celebrated work implies that it has to be flat. This concludes the proof of Theorem A.

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