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Mathematical Analysis

Gromov's dimension comparison problem on Carnot groups

Zoltán M. Balogh [a,1], Jeremy T. Tyson [b,2], Ben Warhurst [c,3]

[a] *Department of Mathematics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland*
[b] *Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, IL 61801, USA*
[c] *School of Mathematics, University of New South Wales, Sydney 2052, Australia*

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Abstract

We solve Gromov's dimension comparison problem on Carnot groups equipped with a Carnot–Carathéodory metric and an adapted Euclidean metric. The proofs use sharp covering theorems relating optimal mutual coverings of Euclidean and Carnot–Carathéodory balls, and elements of sub-Riemannian fractal geometry associated to horizontal self-similar iterated function systems on Carnot groups. ***To cite this article: Z.M. Balogh et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Le problème de dimension comparaison de Gromov sur les groupes de Carnot. Nous présentons la solution du problème de dimension comparaison de Gromov sur les groupes de Carnot muni d'une métrique de Carnot–Carathéodory et une métrique adaptée Euclidienne. Les preuves uilisent des théorèmes de couvrir précises entre des boules Euclidienne et de Carnot–Carathéodory. Nous utilisons aussi des elements de la géométrie fractale sous-Riemanienne associée des fonctions itérées sur les groupes de Carnot. ***Pour citer cet article : Z.M. Balogh et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction and statement of results

Carnot groups are simply connected nilpotent Lie groups with graded Lie algebra equipped with a left-invariant metric of sub-Riemannian type. They arise as ideal boundaries of non-compact rank one symmetric spaces, and serve as both examples of, and local models at regular points for, general sub-Riemannian (Carnot–Carathéodory) manifolds. In his comprehensive survey on intrinsic Carnot–Carathéodory metric geometry [6], Gromov posed the following problem (see Exercise 0.6.C in [6]):

E-mail addresses: zoltan.balogh@math.unibe.ch (Z.M. Balogh), tyson@math.uiuc.edu (J.T. Tyson), warhurst@maths.unsw.edu.au (B. Warhurst).
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Problem 1. Let $(M, \mathcal{H}, g_0)$ be a sub-Riemannian manifold. State and prove sharp comparison results between the Carnot–Carathéodory and Euclidean Hausdorff dimensions of subsets of M.

More precisely, let g be a Riemannian metric on M which tames g_0, and define functions α, β on the power set $\mathcal{P}(M)$ of M as follows: $\alpha(A) = \dim_{(M,g)} A$ and $\beta(A) = \dim_{(M,g_0)} A$, where $\dim_{(M,g)}$ denotes Hausdorff dimension in the metric space (M, g). Determine the range of $\Delta = (\alpha, \beta) : \mathcal{P}(M) \to \mathbb{R}^2$.

Problem 1 asks for a measure-theoretic description of the discrepancy between the sub-Riemannian metric g_0 and any taming Riemannian metric g. Put another way, it asks which Riemannian α-dimensional subsets of M are most nearly horizontal (β is smallest for fixed α) and which are most non-horizontal (β is largest for fixed α). In this note, we describe our solution of Problem 1 in the class of Carnot groups. Full details of the proofs as well as additional results in the context of jet spaces are contained in [4]. This work extends our previous results from [2,3,1] in the particular setting of the Heisenberg group.

Let $\mathbb{G}$ be a Carnot group of dimension at least two with graded Lie algebra $\mathfrak{g} = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_s$ such that $[\mathfrak{v}_1, \mathfrak{v}_i] = \mathfrak{v}_{i+1}$, $i = 1, \ldots, s-1$, and $[\mathfrak{v}_1, \mathfrak{v}_s] = 0$. We equip $\mathfrak{v}_1$ with a left-invariant inner product and denote by d_{cc} the standard sub-Riemannian Carnot–Carathéodory metric defined using this inner product. The Euclidean space underlying $\mathbb{G}$ has dimension $N = \sum_{i=1}^{s} m_i$ while the homogeneous dimension of $\mathbb{G}$ is $Q = \sum_{i=1}^{s} i m_i$, where $\dim \mathfrak{v}_i = m_i$. Our main result is the following statement:

Theorem 1.1. *Let $\mathbb{G}$ be a Carnot group as above. Denote by $\alpha = \dim_E A$, resp. $\beta = \dim_{cc} A$, the Euclidean, resp. Carnot–Carathéodory, Hausdorff dimension of a set $A \subset \mathbb{G}$. Then*

$$\beta_-(\alpha) \leqslant \beta \leqslant \beta_+(\alpha), \tag{1}$$

where $\beta_\pm : [0, N] \to [0, Q]$ are defined as follows.

Let $m_0 = m_{s+1} := 0$. Let $l \in \{0, \ldots, s-1\}$ be the integer satisfying $\sum_{i=0}^{l} m_i < \alpha \leqslant \sum_{i=0}^{l+1} m_i$, then

$$\beta_-(\alpha) = \sum_{i=0}^{l} i m_i + (l+1)\left(\alpha - \sum_{i=1}^{l} m_i\right). \tag{2}$$

Let $q \in \{1, \ldots, s\}$ be the integer satisfying $\sum_{i=q}^{s+1} m_i < \alpha \leqslant \sum_{i=q-1}^{s+1} m_i$, then

$$\beta_+(\alpha) = \sum_{i=q}^{s+1} i m_i + (q-1)\left(\alpha - \sum_{i=q}^{s+1} m_i\right). \tag{3}$$

Let us comment on the formulae for $\beta_-(\alpha)$ and $\beta_+(\alpha)$. Observe that the first component $\sum_{i=0}^{l} i m_i$ in the expression for $\beta_-(\alpha)$ represents the integer part of α weighted against the lowest possible strata in the Lie algebra stratification of $\mathbb{G}$. The second component $(l+1)(\alpha - \sum_{i=1}^{l} m_i)$ is the fractional part of α with weight $l+1$. The function β_+ has a dual interpretation using highest possible strata.

The sharpness of Theorem 1.1 is demonstrated in our next theorem:

Theorem 1.2. *For all $0 \leqslant \alpha \leqslant N$ and $\beta_-(\alpha) \leqslant \beta \leqslant \beta_+(\alpha)$ there exists a set $A_{\alpha,\beta} \subset \mathbb{G}$ with $(\alpha, \beta) = (\dim_E A_{\alpha,\beta}, \dim_{cc} A_{\alpha,\beta})$.*

In Carnot groups, the underlying Euclidean metric plays the role of taming Riemannian metric. Gromov's problem thus admits the solution $\Delta(\mathcal{P}(\mathbb{G})) = \{(\alpha, \beta) \in [0, N] \times [0, Q]:\ \beta_-(\alpha) \leqslant \beta \leqslant \beta_+(\alpha)\}$. Note that since $\beta_\pm$ are monotone increasing and piecewise linear, the planar region $\Delta(\mathcal{P}(\mathbb{G}))$ is a convex polygon. See Example 1 for an explicit description in a specific step three Carnot group, the *Engel group*.

2. Sketch of the proofs

Denote by $\mathcal{H}_E^\alpha$, resp. $\mathcal{H}_{cc}^\beta$ the α-, resp. β-dimensional Hausdorff measures with respect to the Euclidean, resp. CC, metric. Theorem 1.1 is a consequence of the following inequalities relating these measures:

Proposition 2.1. *Let $0 \leqslant \alpha \leqslant N$ and $\beta_\pm(\alpha)$ be as above and let $b > 0$. There exists $L = L(\mathbb{G}, b)$ so that*

$$\mathcal{H}_{cc}^{\beta_+(\alpha)}(A)/L \leqslant \mathcal{H}_E^{\alpha}(A) \leqslant L\mathcal{H}_{cc}^{\beta_-(\alpha)}(A) \tag{4}$$

for all $A \subset B_{cc}(0, b)$, where $B_{cc}(0, R)$ denotes the CC ball of radius R centered at the identity $0 \in \mathbb{G}$.

Proposition 2.1 is established with the aid of the following ball covering lemma:

Lemma 2.2. (a) *For every bounded set $K \subset \mathbb{G}$ and integer $q \in \{2, \ldots, s+1\}$ there exists a constant $M = M(q, K)$ such that every Euclidean ball $B_E(p, r)$ with center $p \in K$ and radius $0 < r < 1$ can be covered by finitely many CC balls $\{B_{cc}(p_j, r^{\frac{1}{q-1}})\}_{j=1}^n$ with $n \leqslant M/r^{\lambda_1(q)}$, where $\lambda_1(q) := \frac{1}{q-1}\sum_{i=q}^{s+1} i m_i - \sum_{i=q}^{s+1} m_i$.*

(b) *For every bounded set $K \subset \mathbb{G}$ and integer $l \in \{0, \ldots, s-1\}$ there exists a constant $M = M(l, K)$ such that every CC ball $B_{cc}(p, r)$ with center $p \in K$ and radius $0 < r < 1$ can be covered by finitely many Euclidean balls $\{B_E(p_j, r^{l+1})\}_{j=1}^n$ with $n \leqslant M/r^{\lambda_2(l)}$, where $\lambda_2(l) := (l+1)\sum_{i=0}^{l} m_i - \sum_{i=0}^{l} i m_i$.*

Observe that $\beta_-(\alpha) = (l+1)\alpha - \lambda_2(l)$ and $\beta_+(\alpha) = (q-1)(\alpha + \lambda_1(q))$, where l and q are as in Theorem 1.1. Parts (a) and (b) of Lemma 2.2 imply the left and right hand inequality in (4) respectively. This concludes the proof of Proposition 2.1 which implies Theorem 1.1.

To prove Theorem 1.2 we note that it suffices to construct sets $A_{\alpha,\beta}$ with $\dim_E A_{\alpha,\beta} = \alpha$ and $\dim_{cc} A_{\alpha,\beta} = \beta$ for $0 \leqslant \alpha \leqslant N$ and only for $\beta = \beta_\pm(\alpha)$. This follows from the monotonicity of Hausdorff dimension and the monotonicity of the functions $\beta_\pm$.

A set A with $\dim_{cc} A = \beta_+(\dim_E A)$ tends to be as *vertical* as possible in that it lies in the direction of higher strata in the Lie algebra. In contrast, $\dim_{cc} A = \beta_-(\dim_E A)$ means that A is as *horizontal* as possible; A lies in the direction of lower strata. Vertical sets are relatively easy to find; horizontal sets are more challenging. The difficulty stems from the non-integrability of the horizontal distribution $\mathfrak{v}_1$. Horizontal sets in two step groups were first constructed by Strichartz [7] as L^∞ graphs. Our approach realizes such sets via fractal geometry, as invariant sets for CC self-similar iterated function systems (IFS).

Proposition 2.3. *Let $\{F_1, \ldots, F_M\}$ be an IFS of contracting similarities of $\mathbb{G}$ with contraction ratios $r_1, \ldots, r_M$, i.e., $d_{cc}(F_i(x), F_i(y)) = r_i d_{cc}(x, y)$ for all $x, y \in \mathbb{G}$ and $1 \leqslant i \leqslant M$. Let $f_i : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1}$ denote the first layer projection of F_i, and assume that the IFS $\{f_1, \ldots, f_M\}$ satisfies the open set condition. Let $\alpha \in (0, m_1]$ be the similarity dimension for the system $\{f_1, \ldots, f_M\}$, e.g., α is the unique solution to the equation $\sum_{i=1}^M r_i^\alpha = 1$. Then there exists $L > 0$ (depending only on α and $\mathbb{G}$) so that*

$$0 < \mathcal{H}_{cc}^{\alpha}(K)/L \leqslant \mathcal{H}_E^{\alpha}(K) \leqslant L\mathcal{H}_{cc}^{\alpha}(K) < \infty,$$

where K denotes the invariant set for the IFS $\{F_1, \ldots, F_M\}$. In particular, $\dim_E K = \dim_{cc} K = \alpha$.

An interesting corollary of Proposition 2.3 is a formula for calculating the dimensions of invariant sets in the Euclidean space underlying $\mathbb{G}$ for a certain class of non-linear IFS, which are not necessarily even generated by Euclidean contractions. Indeed, by the Baker–Campbell–Hausdorff formula, self-similarities of $\mathbb{G}$ are polynomial maps of degree $s-1$. In the Heisenberg group the relevant IFS are generated by affine maps. Dimension formulae for Euclidean self-affine sets have been obtained by Falconer [5]. The following example, the Engel group, is a step three group where the similarities are quadratic polynomials:

Example 1. We model the Engel group with the polynomial group law on $\mathbb{R}^4$ given by $(x_1, x_2, x_3, x_4) \odot (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_2 y_1, x_4 + y_4 + x_3 y_1 + \frac{1}{2} x_2 y_1^2)$. This is a step three Carnot group of homogeneous dimension $Q = 7$. The anisotropic dilation $\delta_s(x_1, x_2, x_3, x_4) = (s x_1, s x_2, s^2 x_3, s^3 x_4)$ is a CC self-similarity with scaling ratio $s > 0$. Consider the *IFS* $\{F_1, F_2, F_3, F_4\}$ given by $F_1(x) = \delta_{1/2}(x)$, $F_2(x) = \delta_{1/2}(p_1 \odot x)$, $F_3(x) = \delta_{1/2}(p_2 \odot x)$, and $F_4(x) = \delta_{1/2}(p_1 \odot p_2 \odot x)$, where $p_1 = (1, 0, 0, 0)$, $p_2 = (0, 1, 0, 0)$. It is clear that projection to the lowest stratum $\mathbb{R}^2$ gives a Euclidean *IFS* satisfying the open set condition whose invariant set is the unit square $[0, 1]^2$. Denoting by Q the invariant set of $\{F_1, F_2, F_3, F_4\}$, Proposition 2.3 gives $\dim_{cc} Q = \dim_E Q = 2$. Note that F_3, F_4 are quadratic maps of $\mathbb{R}^4$, e.g.,

$$F_4(x_1, x_2, x_3, x_4) = (x_1/2 + 1/2, x_2/2 + 1/2, x_1/4 + x_3/4, x_4/8 + x_1^2/16).$$

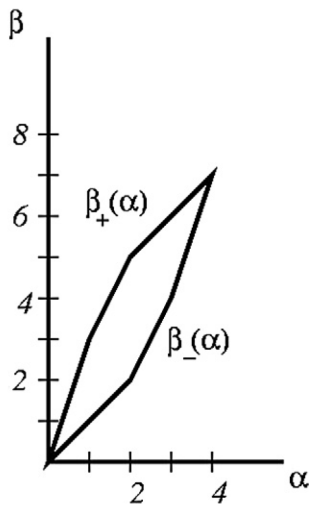

Fig. 1. Dimension plot in the Engel group.

Fig. 1. Plot des dimensions dans la groupe d'Engel.

The full range of Euclidean vs. CC dimensions of subsets of the Engel group is realized by the functions $\beta_\pm : [0,4] \to [0,7]$, $\beta_+(\alpha) = \min\{3\alpha, 2\alpha+1, \alpha+3\}$ and $\beta_-(\alpha) = \max\{\alpha, 2\alpha-2, 3\alpha-5\}$; see Fig. 1.

Proposition 2.3 generates horizontal sets A in the lowest stratum ($0 \leqslant \alpha \leqslant m_1$). Note that in this range $\beta_-(\alpha) = \alpha$. To obtain horizontal sets in higher strata ($m_1 \leqslant \alpha \leqslant N$) as required by Theorem 1.2 we perform an iterative construction starting from a horizontal set A_{m_1} of dimension m_1 and taking successive Carnot products with Cantor-type sets lying in the higher strata. We refer to [3, Theorem 4.1] for a description of the construction in the Heisenberg group.

The Heisenberg and Engel groups model the jet spaces $J^1(\mathbb{R},\mathbb{R})$ and $J^2(\mathbb{R},\mathbb{R})$. Every jet space $J^k(\mathbb{R}^m,\mathbb{R}^n)$ can be equipped with a Carnot group law [8]. We introduce a new Carnot group model for jet spaces in which CC self-similarities are affine maps in the underlying Euclidean geometry, and extend work of Falconer [5] (Euclidean space) and the first two authors [3] (Heisenberg group) on almost sure formulae for CC and Euclidean dimensions of invariant sets for generic representatives in families of self-affine IFS. See Theorems 1.18, 1.19 in [3] for the Heisenberg group case; full details in general jet spaces are in [4].

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