

Probability Theory

An approximation result for nonlinear SPDEs with Neumann boundary conditions

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Abstract

We establish an approximation result to the solution of a semi linear stochastic partial differential equation with a Neumann boundary condition. Our approach is based on the theory of backward doubly stochastic differential equations. **To cite this article:** *N. Mrhardy, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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Résumé

Un résultat d'approximation pour les EDPS avec condition de type Neumann. Nous établissons un résultat d'approximation pour les équations aux dérivées partielles stochastiques nonlinéaires avec conditions de Neumann. Pour ce faire, nous utilisons la théorie des équations différentielles doublement stochastiques rétrogrades. **Pour citer cet article :** *N. Mrhardy, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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1. Introduction and preliminaries

The aim of this Note is to establish an approximation result to the solution of the following Neumann problem:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + Lu(t,x) + f(t,x,u(t,x),\sigma^*(x)\nabla u(t,x)) + g(t,x,u(t,x))\overleftarrow{dB}_t = 0, & (t,x) \in [0,T] \times D, \\ u(T,x) = h(x), & x \in D, \\ \frac{\partial u}{\partial n}(t,x) = 0, & x \in \partial D, \end{cases} \quad (1)$$

where $T > 0$ is a fixed terminal time and L is defined by

$$L = \frac{1}{2} \sum_{i,j=1}^d (\sigma(x)\sigma^*(x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}.$$

Here \overleftarrow{dB}_s denotes the classical backward Itô integral with respect to the Brownian motion B . D is an open, connected and bounded subset of \mathbb{R}^d , f , g , h , b and σ are some measurable functions.

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To put our result in context, we note that, if $g = 0$, the result of this Note is contained in the one of Boufoussi and Casteren [1] giving an approximation result for PDEs with nonlinear Neumann boundary conditions. They used the S -topology to prove a weak convergence result for a sequence of generalized BSDE. This method is difficult to adapt to SPDEs with nonlinear Neumann boundary conditions because of the presence of the backward stochastic integral.

Our approach is based on the connection between backward doubly stochastic differential equations (in short BDSDEs) and stochastic partial differential equations (SPDEs). This topic has been initiated by Pardoux and Peng [6] in the case where the solutions of the SPDEs are regular and extended later by Buckdahn and Ma [3] to the case of stochastic viscosity solutions. Recently, Boufoussi, Mrhardy and van Casteren [2] have given a probabilistic representation to stochastic viscosity solutions of SPDEs with nonlinear Neumann boundary conditions by using the generalized BDSDEs related to reflected diffusions.

1.1. Background

More precisely, we consider two independent d -dimensional Brownian motions ($d \geq 1$), $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ defined on the complete probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ respectively. For any process $(U_s: 0 \leq s \leq T)$ defined on $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ ($i = 1, 2$), we denote $\mathcal{F}_{s,t}^U := \sigma\{U_r - U_s, s \leq r \leq t\}$ and $\mathcal{F}_t^U := \mathcal{F}_{0,t}^U$. Moreover, we define $\Omega \triangleq \Omega_1 \times \Omega_2$, $\mathcal{F} \triangleq \mathcal{F}_1 \otimes \mathcal{F}_2$ and $\mathbb{P} \triangleq \mathbb{P}_1 \otimes \mathbb{P}_2$, and we put $\mathcal{F}_t \triangleq \mathcal{F}_t^W \otimes \mathcal{F}_{t,T}^B \vee \mathcal{N}$, where \mathcal{N} is the collection of \mathbb{P} -null-sets. We notice that the family of σ -algebras $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is not a filtration.

Suppose D is a smooth domain, then one may characterize D and its boundary ∂D by $D = \{\phi > 0\}$, and $\partial D = \{\phi = 0\}$ where ϕ is a twice continuously differentiable and bounded function ($\phi \in C_b^2(\mathbb{R}^d)$) and for all $x \in \partial D$, $\nabla \phi(x)$ is the interior unit normal vector at x . We introduce the function $\rho \in C_b^1(\mathbb{R}^d)$ such that $\rho = 0$ in \bar{D} , $\rho > 0$ in $\mathbb{R}^d \setminus \bar{D}$ and $\rho(x) = (d(x, \bar{D}))^2$ in a neighborhood of \bar{D} such $\langle \nabla \phi(x), \nabla \rho(x) \rangle \leq 0, \forall x \in \mathbb{R}^d$.

Let $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be two globally Lipschitz functions, i.e. for some $K > 0$ and all $x, y \in \mathbb{R}^d$:

$$|b(x) - b(y)| + \|\sigma(x) - \sigma(y)\| \leq K|x - y|.$$

According to Lions and Sznitman [4], for each $x \in \bar{D}$, there exists a unique pair of progressively measurable processes (X_s, k_s) with values in $\bar{D} \times \mathbb{R}_+$ such that

$$\begin{aligned} \text{(i)} \quad X_s^{t,x} &= x + \int_t^s b(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dW_r + \int_t^s \nabla \phi(X_r^{t,x}) dk_r^{t,x}, \quad \text{for } t \leq s \leq T, \\ \text{(ii)} \quad k_t &= \int_0^t I_{\{X_s^{t,x} \in \partial D\}} dk_s^{t,x}, \quad \text{and } k^{t,x} \text{ is increasing.} \end{aligned} \tag{2}$$

Boufoussi et al. [2] have proven, under mild conditions on f and g which we will specify below, that the following BDSDE has an unique solution

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\bar{B}_r - \int_s^T Z_r^{t,x} dW_r. \tag{3}$$

Moreover, $u(t, x) := Y_t^{t,x}$ is a $\mathcal{F}_{t,T}^B$ measurable function which is a stochastic viscosity solution to (1). Then to approach u we are interested to approaching Eq. (3). To this end, we will use an approximation procedure of reflected diffusion $X^{t,x}$ by a penalization method. This method is due to Menaldi [5] and consists in considering the following sequence of SDEs defined by: $n \geq 1$,

$$\begin{aligned} \text{(i)} \quad X_s^{n,t,x} &= x + \int_t^s b(X_r^{n,t,x}) dr + \int_t^s \sigma(X_r^{n,t,x}) dW_r - n \int_t^s \delta(X_s^{n,t,x}) ds, \quad \text{for } t \leq s \leq T, \\ \text{(ii)} \quad x &\in \bar{D}, \end{aligned} \tag{4}$$

where δ is the penalization term defined by $\delta(x) := \nabla \rho(x)$.

Theorem 1.1. Eq. (2) has a unique solution. Moreover, for every $1 \leq p < \infty$,

$$E \left[\sup_{t \leq s \leq T} |X_s^{n,t,x} - X_s^{t,x}|^p \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

and the limit is uniform in $x \in \bar{D}$.

1.2. Assumptions

Let $f : \Omega_2 \times [0, T] \times \bar{D} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, and $g : \Omega_2 \times [0, T] \times \bar{D} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, be two progressively measurable functions with the property that there exist constants $c > 0$ and $0 < \alpha < 1$ such that for any $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \bar{D} \times \mathbb{R} \times \mathbb{R}^d$, the following hypotheses are satisfied:

(H₁) $|f(t, x, y, z) + g(t, x, y, z)| \leq K(1 + |y| + |x| + \|z\|)$.

(H₂) $\begin{cases} |f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)|^2 \leq c(|y_1 - y_2|^2 + |x_1 - x_2|^2 + \|z_1 - z_2\|^2), \\ |g(t, x_1, y_1, z_1) - g(t, x_2, y_2, z_2)|^2 \leq c|y_1 - y_2|^2 + c|x_1 - x_2|^2 + \alpha \|z_1 - z_2\|^2. \end{cases}$

(H₃) Let $h : \bar{D} \rightarrow \mathbb{R}$ be a continuous function, such that for some constant $K > 0$,

$$|h(x)| \leq K(1 + |x|) \quad \forall x \in \bar{D}.$$

2. The main results

For any fixed $n \geq 1$, we consider the following sequence of SPDEs: $\forall (t, x) \in [0, T] \times \mathbb{R}^d$

$$\begin{cases} \frac{\partial u^n(t,x)}{\partial t} + L^n u^n(t, x) + f(t, x, u^n(t, x), \sigma^*(x)\nabla u^n(t, x)) + g(t, x, u^n(t, x)) \overleftarrow{dB}_t = 0, \\ u^n(T, x) = h(x), \end{cases} \tag{5}$$

where $L^n := L - n\delta(x)\nabla$. Let $(X_s^{n,t,x}, s \in [t, T])$ be the unique solution to the SDE (4), it was shown in [3] that $u^n(t, x) := Y_t^{n,t,x}$ is a stochastic viscosity solution to the SPDE (5), where $(Y_s^{n,t,x}, Z_s^{n,t,x})$ is the unique solution the following BDSDE:

$$Y_s^{n,t,x} = h(X_T^{n,t,x}) + \int_s^T f(r, X_r^{n,t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}) dr + \int_s^T g(r, X_r^{n,t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}) \overleftarrow{dB}_r - \int_s^T Z_r^{n,t,x} dW_r.$$

The existence and the uniqueness of the above equation is given in [6]. By using the connection between BDSDEs and SPDEs proved in [2] and [3], we will show that $u^n(t, x)$ converges almost surely, as n goes to infinity, to $u(t, x)$ which is a stochastic viscosity solution of (1). Indeed, we shall assume the following condition:

(H) $g \in C_b^{0,2,3}([0, T] \times \bar{D} \times \mathbb{R}; \mathbb{R}^d)$.

The main result is the following:

Theorem 2.1. Suppose (H₁)–(H₃) and (H) are satisfied. Then, for all $(t, x) \in [0, T] \times \bar{D}$, we have

$$u^n(t, x) \rightarrow u(t, x) \quad \text{as } n \rightarrow \infty, \mathbb{P}\text{-a.s.}$$

Our basic tool for the proof of Theorem 2.1 is the following convergence result:

Theorem 2.2. Under the assumptions (H₁)–(H₃), we have for all $0 \leq t \leq T$

$$E \left(\sup_{t \leq s \leq T} |Y_s^{n,t,x} - Y_s^{t,x}|^2 + \int_t^T \|Z_s^{n,t,x} - Z_s^{t,x}\|^2 ds \right) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Moreover, we have the following convergence:

$$Y_s^{n,t,x} \rightarrow Y_s^{t,x}, \quad \text{as } n \rightarrow \infty, \mathbb{P}\text{-a.s.}$$

Proof. For $n \geq 1$, we put $\Delta Y_s := Y_s^{n,t,x} - Y_s^{t,x}$ and $\Delta Z_s := Z_s^{n,t,x} - Z_s^{t,x}$. By Itô's formula we have

$$\begin{aligned} |\Delta Y_s|^2 + \int_s^T \|\Delta Z_r\|^2 dr &= |h(X_T^{n,t,x}) - h(X_T^{t,x})|^2 + 2 \int_s^T \langle \Delta Y_r, f(r, X_r^{n,t,x}, Y_r^{n,t,x}, Z_s^{n,t,x}) \\ &\quad - f(r, X_r^{t,x}, Y_r^{t,x}, Z_s^{t,x}) \rangle ds + 2 \int_s^T \langle \Delta Y_s, g(s, X_s^{n,t,x}, Y_s^{n,t,x}, Z_s^{n,t,x}) - g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \rangle d\overline{B}_s \\ &\quad + \int_t^T \|g(s, X_s^{n,t,x}, Y_s^{n,t,x}, Z_s^{n,t,x}) - g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\|^2 ds - 2 \int_t^T \langle \Delta Y_s, \Delta Z_s \rangle dW_s. \end{aligned} \quad (6)$$

Taking expectation, we obtain by using assumption (H₂)

$$\mathbb{E} \left(|\Delta Y_s|^2 + \frac{1-\alpha}{6} \int_s^T \|\Delta Z_r\|^2 dr \right) \leq c \mathbb{E} \left[|h(X_T^{n,t,x}) - h(X_T^{t,x})|^2 + \int_t^T |\Delta Y_s^{t,x}|^2 ds + \int_t^T |X_s^{n,t,x} - X_s^{t,x}|^2 ds \right].$$

Applying Gronwall's lemma and Burkholder–Davis–Gundy inequality successively we get

$$\mathbb{E} \left(\sup_{t \leq s \leq T} |\Delta Y_s|^2 + \int_t^T \|\Delta Z_s\|^2 ds \right) \leq c \mathbb{E} \left(|h(X_T^{n,t,x}) - h(X_T^{t,x})|^2 + \sup_{t \leq s \leq T} |X_s^{n,t,x} - X_s^{t,x}|^2 \right).$$

The result is then a consequence of Theorem 1.1 and Lebesgue theorem convergence.

Using similar arguments as before one can show the second assertion. We just take conditional expectation with respect to \mathcal{F}_s in (6), to get

$$|\Delta Y_s|^2 \leq c \mathbb{E}^{\mathcal{F}_s} \left(|h(X_T^{n,t,x}) - h(X_T^{t,x})|^2 + \sup_{0 \leq s \leq T} |X_s^{n,t,x} - X_s^{t,x}|^2 \right).$$

This completes the proof. \square

Remark 1. Notice that in the previous proof we use only conditions (H₁) and (H₂) on function g . In fact, the condition (H) is needed to define the stochastic viscosity solution for SPDEs (5) and (1). For more details in the subject we refer to Boufoussi et al. [2] and Buckdahn and Ma [3].

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