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Statistics

Explicit expressions for moments of t order statistics

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Abstract

The Student's t distribution is the second most popular distribution in statistics, second only to the normal distribution. For the first time, this Note derives explicit closed form expressions for moments of order statistics from the Student's t distribution. ***To cite this article: S. Nadarajah, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Expressions explicites pour des moments de statistiques d'ordre en t. La distribution en t de Student est la seconde loi de distribution la plus utilisée en statistiques après la loi de distribution normale. Pour la première fois cette Note donne, sous forme explicite, les moments de statistiques d'ordre pour la distribution de Student en t. ***Pour citer cet article : S. Nadarajah, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the Student's t distribution with degrees of freedom ν and the probability density function (pdf) given by:

$$f(x) = \frac{1}{\sqrt{\nu} B(1/2, \nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2}, \tag{1}$$

for $-\infty < x < \infty$ and $\nu > 0$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. There has been little work relating to moments of the Student's t order statistics $X_{r:n}$. The only work of which we are aware is Kabir and Rahman [4], where bounds are given for the expected value of $X_{r:n}$.

In this Note, for the first time, we derive expressions for $E(X_{r:n}^k)$ that are finite sums of a known special function – namely, the generalized Kampé de Fériet function (Exton [3], Mathai [5], Aarts [1] and Chaudhry and Zubair [2]) defined by:

$$F_{C:D}^{A:B}\big((a) : (b_1); \ldots; (b_n); (c) : (d_1); \ldots; (d_n); x_1, \ldots, x_n\big)$$

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$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{((a))_{m_1+\cdots+m_n} ((b_1))_{m_1} \cdots ((b_n))_{m_n}}{((c))_{m_1+\cdots+m_n} ((d_1))_{m_1} \cdots ((d_n))_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \tag{2}$$

where $a = (a_1, a_2, \ldots, a_A)$, $b_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,B})$ for $i = 1, 2, \ldots, n$, $c = (c_1, c_2, \ldots, c_C)$, $d_i = (d_{i,1}, d_{i,2}, \ldots, d_{i,D})$ for $i = 1, 2, \ldots, n$, and $((f))_k = ((f_1, f_2, \ldots, f_p))_k = (f_1)_k (f_2)_k \cdots (f_p)_k$ denotes the product of ascending factorials with each ascending factorial defined as $(f_i)_k = f_i(f_i + 1) \cdots (f_i + k - 1)$ with the convention that $(f_i)_0 = 1$. Numerical routines for the direct computation of (2) are available, see Exton [3], Mathai [5], Aarts [1] and Chaudhry and Zubair [2].

2. An explicit expression for $E(X_{r:n}^k)$

If $X_1, X_2, \ldots, X_n$ is a random sample from (1) then it is well known that the pdf of $Z = X_{r:n}$ is given by:

$$f_Z(z) = \frac{n!}{(r-1)!(n-r)!} \{F(z)\}^{r-1} \{1 - F(z)\}^{n-r} f(z)$$

for $r = 1, 2, \ldots, n$, where $F(\cdot)$ is the cumulative distribution function (cdf) corresponding to (1). Since the pdf in (1) is symmetric around zero, the kth moment of Z can be expressed as

$$\begin{aligned}
E(Z^k) &= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} z^k \{F(z)\}^{r-1} \{1 - F(z)\}^{n-r} f(z)\, dz \\
&= \frac{n!}{(r-1)!(n-r)!} \left[\int_0^{\infty} z^k \{F(z)\}^{r-1} \{1 - F(z)\}^{n-r} f(z)\, dz \right. \\
&\quad \left. + (-1)^k \int_0^{\infty} z^k \{F(z)\}^{n-r} \{1 - F(z)\}^{r-1} f(z)\, dz \right] \\
&= A(k, n, r) + (-1)^k A(k, n, n - r + 1).
\end{aligned} \tag{3}$$

Thus, it is sufficient to calculate one of the integrals in (3). Note that $F(\sqrt{w}) = I_{w/(\nu+w)}(1/2, \nu/2)$, where $I_y(\cdot, \cdot)$ denotes the incomplete beta function ratio defined by: $I_y(a, b) = \frac{1}{B(a,b)} \int_0^y w^{a-1}(1 - w)^{b-1}\, dw$. Thus, setting $w = z^2$ and $y = w/(\nu + w)$, one can express $A(k, n, r)$ as

$$\begin{aligned}
A(k, n, r) &= \frac{n!}{2(r-1)!(n-r)!} \int_0^{\infty} w^{(k-1)/2} \{F(\sqrt{w})\}^{r-1} \{1 - F(\sqrt{w})\}^{n-r} f(\sqrt{w})\, dw \\
&= \frac{n!}{2^n (r-1)!(n-r)! \sqrt{\nu} B(1/2, \nu/2)} \int_0^{\infty} w^{(k-1)/2} \left(1 + \frac{w}{\nu}\right)^{-(1+\nu)/2} \\
&\quad \times \left\{1 + I_{w/(\nu+w)}\left(\frac{1}{2}, \frac{\nu}{2}\right)\right\}^{r-1} \left\{1 - I_{w/(\nu+w)}\left(\frac{1}{2}, \frac{\nu}{2}\right)\right\}^{n-r} dw \\
&= \frac{n! \nu^{k/2}}{2^n (r-1)!(n-r)! B(1/2, \nu/2)} \int_0^{\infty} y^{(k-1)/2} (1 - y)^{(\nu-k)/2-1} \\
&\quad \times \left\{1 + I_y\left(\frac{1}{2}, \frac{\nu}{2}\right)\right\}^{r-1} \left\{1 - I_y\left(\frac{1}{2}, \frac{\nu}{2}\right)\right\}^{n-r} dy \\
&= \frac{n! \nu^{k/2}}{2^n (r-1)!(n-r)! B(1/2, \nu/2)} \int_0^{\infty} y^{(k-1)/2} (1 - y)^{(\nu-k)/2-1}
\end{aligned}$$

$$\times \sum_{p=0}^{r-1}\sum_{q=0}^{n-r}\binom{r-1}{p}\binom{n-r}{q}(-1)^q\left\{I_y\left(\frac{1}{2},\frac{\nu}{2}\right)\right\}^{p+q}\mathrm{d}y$$

$$= \frac{n!\nu^{k/2}}{2^n(r-1)!(n-r)!B(1/2,\nu/2)}\sum_{p=0}^{r-1}\sum_{q=0}^{n-r}\binom{r-1}{p}\binom{n-r}{q}(-1)^q$$

$$\times \int_0^\infty y^{(k-1)/2}(1-y)^{(\nu-k)/2-1}\left\{I_y\left(\frac{1}{2},\frac{\nu}{2}\right)\right\}^{p+q}\mathrm{d}y$$

$$= \frac{n!\nu^{k/2}}{2^n(r-1)!(n-r)!B(1/2,\nu/2)}\sum_{p=0}^{r-1}\sum_{q=0}^{n-r}\binom{r-1}{p}\binom{n-r}{q}(-1)^q K(p,q). \tag{4}$$

Using the series expansion,

$$I_x(a,b)=\frac{x^a}{B(a,b)}\sum_{k=0}^{\infty}\frac{(1-b)_k x^k}{(a+k)k!},$$

the integral $K(p,q)$ in (4) can be expressed as

$$K(p,q)=\int_0^1 y^{(k-1)/2}(1-y)^{(\nu-k)/2-1}\left\{\frac{y^{1/2}}{B(1/2,\nu/2)}\sum_{m=0}^{\infty}\frac{(1-\nu/2)_m y^m}{(1/2+m)m!}\right\}^{p+q}\mathrm{d}y$$

$$=\int_0^1\sum_{m_1=0}^{\infty}\cdots\sum_{m_{p+q}=0}^{\infty}\frac{(1-\nu/2)_{m_1}\cdots(1-\nu/2)_{m_{p+q}}}{B^{p+q}(1/2,\nu/2)(1/2+m_1)\cdots(1/2+m_{p+q})m_1!\cdots m_{p+q}!}$$

$$\times y^{(k+p+q-1)/2+m_1+\cdots+m_{p+q}}(1-y)^{(\nu-k)/2-1}\,\mathrm{d}y$$

$$=\sum_{m_1=0}^{\infty}\cdots\sum_{m_{p+q}=0}^{\infty}\frac{(1-\nu/2)_{m_1}\cdots(1-\nu/2)_{m_{p+q}}}{B^{p+q}(1/2,\nu/2)(1/2+m_1)\cdots(1/2+m_{p+q})m_1!\cdots m_{p+q}!}$$

$$\times\int_0^1 y^{(k+p+q-1)/2+m_1+\cdots+m_{p+q}}(1-y)^{(\nu-k)/2-1}\,\mathrm{d}y$$

$$=\sum_{m_1=0}^{\infty}\cdots\sum_{m_{p+q}=0}^{\infty}\frac{(1-\nu/2)_{m_1}\cdots(1-\nu/2)_{m_{p+q}}}{B^{p+q}(1/2,\nu/2)(1/2+m_1)\cdots(1/2+m_{p+q})m_1!\cdots m_{p+q}!}$$

$$\times B\left(\frac{k+p+q+1}{2}+m_1+\cdots+m_{p+q},\frac{\nu-k}{2}\right). \tag{5}$$

Using the fact $(f)_k=\Gamma(f+k)/\Gamma(f)$ and the definition in (2), one can reexpress (5) as

$$K(p,q)=2^{p+q}B^{-p-q}(1/2,\nu/2)B\left(\frac{\nu-k}{2},\frac{k+p+q+1}{2}\right)$$

$$\times F_{1:1}^{1:2}\left(\left(\frac{k+p+q+1}{2}\right):\left(1-\frac{\nu}{2},\frac{1}{2}\right);\ldots;\left(1-\frac{\nu}{2},\frac{1}{2}\right);\right.$$

$$\left.\left(\frac{\nu+p+q+1}{2}\right):\left(\frac{3}{2}\right);\ldots;\left(\frac{3}{2}\right);1,\ldots,1\right). \tag{6}$$

Combining (4) and (6), we obtain the expression:

$$A(k,n,r)=\frac{n!\nu^{k/2}}{2^n(r-1)!(n-r)!}\sum_{p=0}^{r-1}\sum_{q=0}^{n-r}\binom{r-1}{p}\binom{n-r}{q}(-1)^q 2^{p+q}$$

$$
\begin{aligned}
&\times B^{-1-p-q}(1/2,\nu/2)B\left(\frac{\nu-k}{2},\frac{k+p+q+1}{2}\right)\\
&\times F^{1:2}_{1:1}\left(\left(\frac{k+p+q+1}{2}\right):\left(1-\frac{\nu}{2},\frac{1}{2}\right);\ldots;\left(1-\frac{\nu}{2},\frac{1}{2}\right);\right.\\
&\left.\left(\frac{\nu+p+q+1}{2}\right):\left(\frac{3}{2}\right);\ldots;\left(\frac{3}{2}\right);1,\ldots,1\right). \qquad (7)
\end{aligned}
$$

Note that (7) is a finite sum of the generalized Kampé de Fériet function. It is easy to see that the infinite sum in (5) converges if $k<\nu$ and hence that the expression given by (7) exists for $k<\nu$. For instance, if $\nu/2>k/2$ is an integer then (5) will be a finite sum. If $\nu/2>k/2$ is a non-integer then one can have $B((k+p+q+1)/2+m_1+\cdots+m_{p+q},(\nu-k)/2)<\Gamma((\nu-k)/2)$ for all sufficiently large $m_1+\cdots+m_{p+q}$. Thus, for a sufficiently large N, one can obtain a bound

$$
\begin{aligned}
&\left|\sum_{\max(m_1,\ldots,m_{p+q})>N}\cdots\sum\frac{(1-\nu/2)_{m_1}\cdots(1-\nu/2)_{m_{p+q}}}{B^{p+q}(1/2,\nu/2)(1/2+m_1)\cdots(1/2+m_{p+q})m_1!\cdots m_{p+q}!}\right.\\
&\quad\left.\times B\left(\frac{k+p+q+1}{2}+m_1+\cdots+m_{p+q},\frac{\nu-k}{2}\right)\right|\\
&<\frac{\Gamma((\nu-k)/2)}{B^{p+q}(1/2,\nu/2)}\sum_{\max(m_1,\ldots,m_{p+q})>N}\cdots\sum\frac{|(1-\nu/2)_{m_1}\cdots(1-\nu/2)_{m_{p+q}}|}{(1/2+m_1)\cdots(1/2+m_{p+q})m_1!\cdots m_{p+q}!}\\
&=\frac{\Gamma((\nu-k)/2)}{B^{p+q}(1/2,\nu/2)}\left[\sum_{m_1=0}^{\infty}\cdots\sum_{m_{p+q}=0}^{\infty}\frac{|(1-\nu/2)_{m_1}\cdots(1-\nu/2)_{m_{p+q}}|}{(1/2+m_1)\cdots(1/2+m_{p+q})m_1!\cdots m_{p+q}!}\right.\\
&\quad\left.-\sum_{m_1=0}^{N}\cdots\sum_{m_{p+q}=0}^{N}\frac{|(1-\nu/2)_{m_1}\cdots(1-\nu/2)_{m_{p+q}}|}{(1/2+m_1)\cdots(1/2+m_{p+q})m_1!\cdots m_{p+q}!}\right]\\
&=\frac{\Gamma((\nu-k)/2)}{B^{p+q}(1/2,\nu/2)}\left[\left\{\sum_{m=0}^{\infty}\frac{|(1-\nu/2)_m|}{(1/2+m)m!}\right\}^{p+q}-\left\{\sum_{m=0}^{N}\frac{|(1-\nu/2)_m|}{(1/2+m)m!}\right\}^{p+q}\right]<\infty.
\end{aligned}
$$

Note that the existence of the ordinary moments of (1) also require the condition $k<\nu$.

3. Conclusions

We have derived an expression for moments of Student's t order statistics as a finite sum of a known special function. This expression is the first known one in explicit and exact form.

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