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Partial Differential Equations

Macroscopic limit of self-driven particles with orientation interaction

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Abstract

The discrete Couzin–Vicsek algorithm (CVA) has been proposed to model the interactions of individuals among animal societies such as schools of fish. In this Note, we propose a kinetic (mean-field) version of the CVA model and provide its formal macroscopic limit. The final macroscopic model involves a conservation equation for the density of the individuals and a non-conservative equation for the director of the mean velocity. The result is based on the introduction of a non-conventional concept of a collisional invariant of the collision operator. ***To cite this article: P. Degond, S. Motsch, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Limite macroscopique de particules autopropulsées avec interaction d'orientation. L'algorithme discret de Couzin–Vicsek (CVA) a été proposé pour modéliser l'interaction d'individus au sein de sociétés animales comme les bancs de poissons. Dans cette Note, nous proposons une version cinétique (champ-moyen) de l'algorithme CVA et en donnons la limite macroscopique formelle. Le modèle macroscopique final comprend une équation de conservation pour la densité des individus et une équation non-conservative pour le vecteur directeur de la vitesse moyenne. Ce résultat est basé sur l'introduction d'un concept non-conventionnel d'invariant collisionnel de l'opérateur de collision. ***Pour citer cet article : P. Degond, S. Motsch, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Le modèle discret de Couzin–Vicsek (CVA) [1–3,8,16] a été proposé pour modéliser les *interactions entre individus* au sein de *sociétés animales* telles que les bancs de poissons. Les individus se déplacent à vitesse constante. Le modèle CVA décrit l'évolution *discrète en temps* des positions des individus et de l'angle de leur vitesse mesuré à partir d'un axe de référence. A chaque pas de temps, l'angle est mis à jour à une nouvelle valeur donnée par la *direction de la vitesse moyenne des particules voisines*, avec l'addition d'un *bruit*. Les positions sont augmentées de la distance parcourue durant le pas de temps par la particule dans la direction spécifiée par l'angle de la vitesse.

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Pour modéliser des *bancs de poissons* pouvant atteindre plusieurs millions d'individus, il est légitime de chercher des *modèles continus*, qui décrivent la société par des *variables macroscopiques* telles que la densité, la vitesse moyenne, etc. Plusieurs modèles de ce type existent [10,14,15] mais ils sont phénoménologiques et non basés sur un établissement rigoureux à partir des interactions individuelles. Plusieurs tentatives pour établir des modèles continus à partir du modèle CVA ont été proposées dans la littérature [9,12,13] mais certaines étapes de la dérivation demandent à être précisées et les qualités mathématiques des modèles obtenus n'ont pas encore été analysées. On pourra également se référer à [6,11] pour des modèles connexes. Un autre modèle a été proposé dans [7] sur la base de mesures expérimentale, et sa dynamique macroscopique est l'objet de [4].

Cette Note constitue la première tentative (à notre connaissance) pour établir une *version continue* du modèle CVA à partir d'une *limite macroscopique* d'une *formulation cinétique* de ce modèle. Elle résume un travail en cours [5]. A cet effet, nous proposons une *formulation cinétique* du modèle CVA (Éqs. (1), (2)). Ce modèle dépend d'un *petit paramètre* ε qui décrit le passage à l'*échelle macroscopique*. Lorsque $\varepsilon \to 0$, nous sommes en mesure de montrer formellement que la fonction de distribution f^ε solution de (1), (2) converge vers une *distribution d'équilibre* dépendant paramétriquement de la *densité* $\rho(x,t)$ et du *vecteur directeur du flux* $\Omega(x,t)$. A la limite, les fonctions ρ et Ω évoluent selon un système d'*équations macroscopiques* donné par (5), (6) (voir Théorème 2.1). Dans [5], il est prouvé que ce modèle est *hyperbolique*.

La démonstration du Théorème 2.1 repose sur l'étude de l'*opérateur de collision* du modèle cinétique. Le Lemme 3.1 précise la forme des *équilibres* et en particulier, leur dépendance par rapport à la densité ρ et au vecteur directeur du flux Ω. L'outil principal pour prouver ce lemme est l'inégalité (8) qui exprime une forme de *dissipation d'entropie*. Le point le plus original de ce travail concerne l'introduction d'*invariants collisionnels généralisés*. En effet, l'opérateur de collision n'a pas d'autre invariant collisionnel au sens classique que les fonctions constantes, ce qui ne suffit pas pour déterminer le modèle limite. Nous définissons des invariants collisionnels généralisés à l'Éq. (10) ou (11). Ils dépendent d'une *direction arbitraire* et s'appliquent aux fonctions de distribution ayant un *flux parallèle à cette direction*. Ils ne sont pas connus explicitement mais le Lemme 3.2 permet de montrer que l'espace des invariants collisionnels généralisés (pour une direction de flux donnée) est un espace de *dimension 3* (Proposition 3.1). Ils dépendent en fait d'une même fonction scalaire $g(\mu)$ de la variable $\mu \in (-1,1)$ solution du *problème elliptique* (13). Une fois ces invariants collisionnels déterminés, le modèle macroscopique s'obtient en intégrant le modèle cinétique (1) contre ces invariants collisionnels et en faisant tendre ε vers 0. Les calculs sont détaillés dans [5] et fournissent le modèle (5), (6).

1. Introduction

The discrete Couzin–Vicsek algorithm (CVA) [1–3,8,16] has been proposed to model the interactions of individuals among animal societies such as schools of fish. The individuals move with a velocity of constant magnitude. The CVA model describes the time-discrete evolution of the positions of the individuals and of the velocity angle measured from a reference direction. At each tick of the clock, the angle is updated to a new value given by the director of the average velocity of the neighbouring particles, with addition of noise. The positions are updated by adding the distance travelled during the time step by the fish in the direction specified by its velocity angle.

For the modelling of large schools of fish which can reach up to several million individuals, it is meaningful to look for continuum like models, which describe the fish society by macroscopic variables such as the mean density, the mean velocity and so on. Several such models exist (see e.g. [10,14,15]) but they are phenomenological and not based on a 'rigorous' derivation from individual interactions. Several attempts to derive continuum models from the CVA model are also reported in the literature [9,12,13], but some steps in the derivation would need further inspection and the mathematical 'qualities' of the resulting models have not been analysed yet. One can also refer to [6,11] for related models. An alternate model has been proposed in [7] on the basis of experimental measurements, and its large-scale dynamics is studied in [4].

This Note is the first attempt (to our knowledge) to base the derivation of a continuum version of the CVA model on a macroscopic limit of a kinetic equation. To this aim, we propose a kinetic (mean-field) version of the CVA model. Its collision operator has a three-dimensional manifold of equilibria but only a one-dimensional manifold of collisional invariants. To find the evolution of the macroscopic variables, we need to weaken the concept of a collisional invariant (as opposed to conventional kinetic theory). We believe this concept can be useful for other macroscopic limits presenting similar features.

This Note provides a summary of the matter developed in [5]. It is organised as follows: in Section 2, the kinetic version of the CVA model is introduced and the main theorem (Theorem 2.1) is stated. Section 3 summarises the properties of the collision operator while Section 4 gives an outline of the formal convergence proof itself.

2. A kinetic (mean-field) version of the CVA model and its macroscopic limit

We consider the following kinetic model for a distribution function $f^\varepsilon(x,\omega,t)$, where $x \in \mathbb{R}^3$ is the particle position, $\omega \in \mathbb{S}^2$ is its velocity, belonging to the unit two-dimensional sphere, and $t \geqslant 0$ is the time:

$$\varepsilon(\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon) = -\nabla_\omega \cdot (F^\varepsilon f^\varepsilon) + d\Delta_\omega f^\varepsilon, \qquad F^\varepsilon(x,\omega,t) = \nu(\omega \cdot \bar{\omega}^\varepsilon)(\mathrm{Id} - \omega \otimes \omega)\bar{\omega}^\varepsilon(x,\omega,t), \tag{1}$$

$$\bar{\omega}^\varepsilon(x,\omega,t) = J[f^\varepsilon]/|J[f^\varepsilon]|, \qquad J[f^\varepsilon] = \int\limits_{y\in\mathbb{R}^3, \upsilon\in\mathbb{S}^2} K\big(|(x-y)/\varepsilon|\big)\upsilon f^\varepsilon(y,\upsilon,t)\,\mathrm{d}y\,\mathrm{d}\upsilon. \tag{2}$$

The left-hand side of the first Eq. (1) models the spatial motion of the particles while the right-hand side combines the action of the interaction force F^ε and of a Gaussian noise in velocity with diffusion coefficient d. The interaction force (second Eq. (1)) describes the relaxation at rate ν of the particle velocity ω towards the director $\bar{\omega}^\varepsilon$ of the total particle flux $J[f^\varepsilon]$ in a small neighbourhood specified by the kernel K (see (2); typically, K is the indicator function of a ball of radius R). The model is already presented in scaled form i.e. all variables are dimensionless and of typical magnitude unity, except ε which represents the ratio of the actual particle mean interaction time to the typical observation time. Looking for a macroscopic limit means that the observation time is much longer than the mean interaction time, i.e. $\varepsilon \ll 1$. That this model provides a kinetic version of the CVA model is developed in [5].

By Taylor expansion with respect to ε, the interaction force can be expanded as follows:

$$F^\varepsilon = F[f^\varepsilon] + \mathrm{O}(\varepsilon^2), \qquad F[f^\varepsilon](x,\omega,t) = \nu(\omega \cdot \Omega^\varepsilon)(\mathrm{Id} - \omega \otimes \omega)\Omega^\varepsilon(x,t), \tag{3}$$

$$\Omega^\varepsilon(x,t) = \frac{j^\varepsilon(x,t)}{|j^\varepsilon(x,t)|}, \quad \text{and} \quad j^\varepsilon(x,t) = \int\limits_{\upsilon\in\mathbb{S}^2} \upsilon f^\varepsilon(x,\upsilon,t)\,\mathrm{d}\upsilon. \tag{4}$$

The quantity $j^\varepsilon(x,t)$ is the particle flux. The density is defined by $\rho^\varepsilon(x,t) = \int_{\upsilon\in\mathbb{S}^2} f^\varepsilon(x,\upsilon,t)\,\mathrm{d}\upsilon$. We note that observing the system at large scales makes the interaction local and that this interaction tends to align the particle velocity to the direction of the local particle flux. This interaction term is balanced at leading order by the diffusion term which tends to spread the particles isotropically on the sphere. Obviously, an equilibrium distribution results from the balance of these two antagonist phenomena. The continuum model describes the dynamics along this equilibrium manifold. More precisely, the formal limit as $\varepsilon \to 0$ of the model is given by the following theorem (which is the main goal of the paper):

Theorem 2.1. *The formal limit as $\varepsilon \to 0$ of f^ε is given by $f^0 = \rho M_\Omega$ where $\rho(x,t) = \int_{\omega\in\mathbb{S}^2} f^0(x,\omega,t)\,\mathrm{d}\omega$ is the total mass, $\Omega(x,t) = j(x,t)/|j(x,t)| \in \mathbb{S}^2$ is the director of the flux $j(x,t) = \int_{\omega\in\mathbb{S}^2} f^0(x,\omega,t)\omega\,\mathrm{d}\omega$, and M_Ω is a given function of $\omega \cdot \Omega$ only depending on ν and d which will be specified later on* (*see* (7))*. Furthermore, $\rho(x,t)$ and $\Omega(x,t)$ satisfy the following system of first order partial differential equations*:

$$\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega) = 0. \tag{5}$$

$$\rho\big(\partial_t \Omega + c_2(\Omega \cdot \nabla)\Omega\big) + \lambda(\mathrm{Id} - \Omega \otimes \Omega)\nabla_x \rho = 0, \tag{6}$$

where the convection speeds c_1, c_2 and the interaction constant λ will be specified in the course of the paper (*see Section* 4)*.*

In [5], it is proved that the two-dimensional version of this model is hyperbolic, and thus, at least locally well-posed. This theorem is based on the study of the operator $Q(f) = -\nabla_\omega \cdot (F[f]f) + d\Delta_\omega f$, with $F[f]$ defined by (3), which is developed in the next section.

3. Study of the collision operator Q

We begin by looking for the equilibrium solutions, i.e. the functions f which cancel Q. Let $\mu = \cos\theta$. We denote by $\sigma(\mu)$ an antiderivative of $\nu(\mu)$, i.e. $(\mathrm{d}\sigma/\mathrm{d}\mu)(\mu) = \nu(\mu)$. We define

$$M_\Omega(\omega) = C\exp\bigl(\sigma(\omega\cdot\Omega)/d\bigr), \quad \text{where } C \text{ is such that } \int M_\Omega(\omega)\,\mathrm{d}\omega = 1. \tag{7}$$

Lemma 3.1. (i) *The operator Q can be written as $Q(f) = d\nabla_\omega\cdot[M_{\Omega[f]}\nabla_\omega(f/M_{\Omega[f]})]$, and we have*

$$H(f) := \int_{\omega\in\mathbb{S}^2} Q(f)\frac{f}{M_{\Omega[f]}}\,\mathrm{d}\omega = -d\int_{\omega\in\mathbb{S}^2} M_{\Omega[f]}\left|\nabla_\omega\left(\frac{f}{M_{\Omega[f]}}\right)\right|^2\mathrm{d}\omega \leqslant 0. \tag{8}$$

(ii) *The equilibria, i.e. the functions $f(\omega)$ such that $Q(f) = 0$ form a three-dimensional manifold $\mathcal{E}$ given by $\mathcal{E} = \{\rho M_\Omega(\omega) \mid \rho\in\mathbb{R}_+,\ \Omega\in\mathbb{S}^2\}$, and ρ is the total mass while Ω is the director of the flux of $\rho M_\Omega(\omega)$, i.e.*

$$\int_{\omega\in\mathbb{S}^2}\rho M_\Omega(\omega)\,\mathrm{d}\omega = \rho, \qquad \Omega = j[\rho M_\Omega]/\bigl|j[\rho M_\Omega]\bigr|, \qquad j[\rho M_\Omega] = \int_{\omega\in\mathbb{S}^2}\rho M_\Omega(\omega)\omega\,\mathrm{d}\omega.$$

Furthermore, $H(f) = 0$ if and only if $f = \rho M_\Omega$ for arbitrary $\rho\in\mathbb{R}_+$ and $\Omega\in\mathbb{S}^2$.

This theorem is proved in [5]. The flux can be written $j[\rho M_\Omega] = \langle\cos\theta\rangle_M\rho\Omega$, where for any function $g(\cos\theta)$, the symbol $\langle g(\cos\theta)\rangle_M$ denotes the average of g over the probability distribution M_Ω, i.e.

$$\bigl\langle g(\cos\theta)\bigr\rangle_M = \int M_\Omega(\omega)g(\omega\cdot\Omega)\,\mathrm{d}\omega = \frac{\int_0^\pi g(\cos\theta)\exp(\frac{\sigma(\cos\theta)}{d})\sin\theta\,\mathrm{d}\theta}{\int_0^\pi \exp(\frac{\sigma(\cos\theta)}{d})\sin\theta\,\mathrm{d}\theta}. \tag{9}$$

The collision invariants of Q are the functions $\psi(\omega)$ s.t. $\int_{\omega\in\mathbb{S}^2} Q(f)\psi\,\mathrm{d}\omega = 0,\ \forall f$. Clearly, constant functions are collision invariants, but there is no other obvious conservation relation, since momentum is not conserved by the interaction operator. We slightly weaken the concept of a collision invariant in the following way. We fix $\Omega\in\mathbb{S}^2$ arbitrarily, and we look for all ψ's which are collisional invariants of $Q(f)$ for all f with director $\Omega[f] = \Omega$. This is equivalent to saying that $j[f]$ is aligned with $\Omega[f]$, or

$$0 = \Omega\times j[f] = \int_{\omega\in\mathbb{S}^2} f(\Omega\times\omega)\,\mathrm{d}\omega.$$

This last formula can be viewed as a linear constraint and, introducing the Lagrange multiplier β of this constraint, β being a vector normal to Ω, we can restate the problem of finding the 'generalised' collisional invariants as follows: Given $\Omega\in\mathbb{S}^2$, find all ψ's such that there exist $\beta\in\mathbb{R}^3$ with $\Omega\cdot\beta = 0$, and

$$\int_{\omega\in\mathbb{S}^2}\frac{f}{M_\Omega}\bigl\{\nabla_\omega\cdot(M_\Omega\nabla_\omega\psi) - \beta\cdot(\Omega\times\omega)M_\Omega\bigr\}\,\mathrm{d}\omega = 0, \quad \forall f. \tag{10}$$

Now, (10) holds for all f without constraint and immediately leads to the following problem for ψ:

$$\nabla_\omega\cdot(M_\Omega\nabla_\omega\psi) = \beta\cdot(\Omega\times\omega)M_\Omega. \tag{11}$$

The set $\mathcal{C}_\Omega$ of generalised collisional invariants associated with the vector Ω is a vector space. It is convenient to introduce a Cartesian basis (e_1, e_2, Ω) and the associated spherical coordinates (θ,ϕ). Then $\beta = (\beta_1,\beta_2,0)$ and $\beta\cdot(\Omega\times\omega) = (-\beta_1\sin\phi + \beta_2\cos\phi)\sin\theta$. Therefore, we can successively solve for ψ_1 and ψ_2, the solutions of (11) with right-hand sides respectively equal to $-\sin\phi\sin\theta M_\Omega$ and $\cos\phi\sin\theta M_\Omega$. We are looking for solutions in an $L^2(\mathbb{S}^2)$ framework. The following lemma is an elementary application of Lax–Milgram's theorem:

Lemma 3.2. *Let $\chi\in L^2(\mathbb{S}^2)$ such that $\int\chi\,\mathrm{d}\omega = 0$. The problem*

$$\nabla_\omega\cdot(M_\Omega\nabla_\omega\psi) = \chi, \tag{12}$$

has a unique weak solution in the space $\mathring{H}^1(\mathbb{S}^2)$*, the quotient of the space* $H^1(\mathbb{S}^2)$ *by the space spanned by the constant functions, endowed with the quotient norm.*

So, to each of the right-hand sides $\chi = -\sin\phi \sin\theta M_\Omega$ or $\chi = \cos\phi \sin\theta M_\Omega$ which have zero average on the sphere, there exist solutions ψ_1 and ψ_2 respectively (unique up to constants) of problem (12). We single out unique solutions by requesting that ψ_1 and ψ_2 have zero average on the sphere: $\int \psi_k \, \mathrm{d}\omega = 0$, $k = 1, 2$. We can state the following corollary to Lemma 3.2:

Proposition 3.1. *The set* $\mathcal{C}_\Omega$ *of generalised collisional invariants associated with the vector* Ω *which belong to* $H^1(\mathbb{S}^2)$ *is a three dimensional vector space* $\mathcal{C}_\Omega = \mathrm{Span}\{1, \psi_1, \psi_2\}$.

More explicit forms for ψ_1 and ψ_2 can be found. By expanding in Fourier series with respect to ϕ, we easily see that $\psi_1 = -g(\cos\theta)\sin\phi$, $\psi_2 = g(\cos\theta)\cos\phi$, where $g(\mu)$ is the unique solution of the elliptic problem on $[-1, 1]$:

$$-(1-\mu^2)\partial_\mu\big(\mathrm{e}^{\sigma(\mu)/d}(1-\mu^2)\partial_\mu g\big) + \mathrm{e}^{\sigma(\mu)/d} g = -(1-\mu^2)^{3/2}\mathrm{e}^{\sigma(\mu)/d}. \tag{13}$$

We note that no boundary condition is needed to specify g uniquely since the operator at the left-hand side of (13) is degenerate at the boundaries $\mu = \pm 1$. Indeed, it is an easy matter, using again Lax–Milgram theorem, to prove that problem (13) has a unique solution in the weighted H^1 space V defined by $V = \{g \mid (1-\mu^2)^{-1/2} g \in L^2(-1, 1),\ (1-\mu^2)^{1/2}\partial_\mu g \in L^2(-1, 1)\}$. Furthermore, the Maximum Principle shows that g is non-positive. For convenience, we introduce $h(\mu) = (1-\mu^2)^{-1/2} g \in L^2(-1, 1)$ or equivalently $h(\cos\theta) = g(\cos\theta)/\sin\theta$. We then define $\Psi(\omega) = (\Omega \times \omega) h(\Omega \cdot \omega) = \psi_1 e_1 + \psi_2 e_2$. Ψ is the vector generalised collisional invariant associated with the direction Ω.

4. Formal limit as $\varepsilon \to 0$

The goal of this section is to prove Theorem 2.1. We suppose that all functions are as regular as needed and that all convergences are as strong as needed. The rigorous proof of this convergence result seems to present considerable challenges.

We start with Eq. (1) and we suppose that $f^\varepsilon \to f$ when $\varepsilon \to 0$. Then, from (1), $Q(f^\varepsilon) = \mathrm{O}(\varepsilon)$ and we deduce that $Q(f) = 0$. By Lemma 3.1, $f = \rho M_\Omega$, with $\rho \geqslant 0$ and $\Omega \in \mathbb{S}^2$. To find how ρ and Ω depend on (x, t), we use the generalised collisional invariants. First, the constant collisional invariants obviously lead to the continuity equation (5) with $c_1 = \langle \cos\theta \rangle_M$. Now, we multiply (1) by $\Psi^\varepsilon = h(\omega \cdot \Omega[f^\varepsilon])(\Omega[f^\varepsilon] \times \omega)$, integrate with respect to ω and take the limit as $\varepsilon \to 0$. We note that $\Omega[f^\varepsilon] \to \Omega$ and that Ψ^ϵ is smooth enough (given the functional spaces used for the existence theory), and consequently, $\Psi^\varepsilon \to \Psi = h(\omega \cdot \Omega)(\Omega \times \omega)$. Therefore, in the limit as $\varepsilon \to 0$, we get:

$$\Omega \times X = 0, \quad X := \int_{\omega \in \mathbb{S}^2} \big(\partial_t(\rho M_\Omega) + \omega \cdot \nabla_x(\rho M_\Omega)\big) h(\omega \cdot \Omega)\,\omega \, \mathrm{d}\omega. \tag{14}$$

The explicit integration of the right-hand side of (14) is detailed in [5] and leads to (6) with $c_2 = \langle \cos\theta \rangle_{(\sin^2\theta)\nu h M}$, $\lambda = d\langle 1/\nu \rangle_{(\sin^2\theta)\nu h M}$, where the brackets $\langle \cdot \rangle_{(\sin^2\theta)\nu h M}$ are defined in a similar way as (9), replacing M_Ω by $(\sin^2\theta)\nu h M_\Omega$.

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