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Differential Geometry

Positively curved π_2-finite manifolds

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Abstract

Let M be a smooth manifold with finite second homotopy group, positive sectional curvature, dimension greater than 8, and assume that a compact connected Lie group G acts smoothly on M. We prove the vanishing of the characteristic number $\hat{A}(M, TM)$ if G contains two commuting involutions. ***To cite this article: H. Herrera, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Variétés avec π_2 fini et courbure positive. Soit M une variété lisse avec un deuxième groupe d'homotopie fini, de courbure sectionnelle positive et de dimension plus grande que 8. Soit G un groupe de Lie compact et connexe qui agit de façon C^∞ sur M. On démontre que le nombre caractéristique $\hat{A}(M, TM)$ s'annule si G contient deux involutions qui commutent entre elles. ***Pour citer cet article : H. Herrera, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

The $\hat{A}$-genus provides a characteristic number which is the obstruction to the existence of certain geometrical structures. For instance, Lichnerowicz showed that, for a closed Riemannian Spin manifold M with positive scalar curvature, $\hat{A}(M) = 0$. Hence, the $\hat{A}$-genus is an obstruction to the existence of positive scalar curvature metrics on $4n$-dimensional Spin manifolds. Atiyah and Hirzebruch proved the vanishing of the $\hat{A}$-genus on Spin manifolds admitting a non-trivial smooth circle action. This vanishing was generalized in [4] to *non-Spin manifolds with finite second homotopy group*. Thus the $\hat{A}$-genus is an obstruction to the existence of Lie group actions on Spin manifolds and on π_2-finite manifolds.

In this Note we establish $\hat{A}(M, T) = \langle \hat{\mathcal{A}}(M) \cdot \mathrm{ch}(T), [M] \rangle$ as an obstruction to positive sectional curvature on non-Spin π_2-finite manifolds under certain assumptions on the symmetries. Here $\mathrm{ch}(T)$ denotes the Chern class of the complexified tangent bundle $T = TM \otimes \mathbb{C}$. This work is an application of the rigidity theorem proved in [4], and follows [2] closely.

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In dimension 4, there is only one characteristic number $\hat{A}(M)$, since all others can be written in terms of it. In particular $\hat{A}(M,T) = -20\hat{A}(M)$. In dimension 8, there are two Pontrjagin numbers given by the classes $p_1(M)^2$ and $p_2(M)$. In this case we have that $\mathbb{HP}^2$ admits a metric with positive sectional curvature, $\hat{A}(\mathbb{HP}^2) = 0$, and $\hat{A}(\mathbb{HP}^2, T\mathbb{HP}^2_c) = -1 \neq 0$. In dimension greater than or equal to 12 no counterexample is known, preventing $\hat{A}(M,T)$ from becoming the obstruction to positive sectional curvature.

Despite the fact that the characteristic number $\hat{A}(M,T)$ may not be an integer and it is not the index of a Dirac operator on non-Spin manifolds, we can make use of it by means of the elliptic genus as in [4]. The main theorem of this Note is the following:

Theorem 1.1. *Let M be a closed connected π_2-finite manifold of dimension greater than* 8. *Suppose M admits a metric of positive sectional curvature and a smooth action by a compact connected Lie group G. Furthermore, assume that there is a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset G$ acting effectively and isometrically on M. Then $\hat{A}(M) = 0$ and $\hat{A}(M,T) = 0$.*

Recall that the signature of a 12-dimensional manifold M is given by $\operatorname{sign}(M) = 8\hat{A}(M,T) - 32\hat{A}(M)$. Thus, we can immediately see that if M is 12-dimensional and satisfies the hypothesis of the theorem, then $\operatorname{sign}(M) = 0$. Manifolds with finite second homotopy group have been considered in the context of bounded sectional curvature in [7], elliptic genera and quaternion-Kähler manifolds [4].

2. π_2-finite manifolds with S^1 actions

Let M be an oriented, connected, compact $2n$-dimensional manifold. We say that a manifold is π_2-*finite* if $|\pi_2(M)| < \infty$.

Assume M is endowed with a (non-trivial) smooth S^1-action. Let M^{S^1} denote the fixed point set of the circle action. At each point $p \in M^{S^1}$, the tangent space of M splits as a sum of S^1 representations, $T_pM = T_pM^{S^1} \oplus L^{m_1} \oplus \cdots \oplus L^{m_k}$, where L^a denotes the S^1 representation on which $\lambda \in S^1$ acts by multiplication by λ^a. The space $T_pM^{S^1}$ is a trivial representation of S^1. The numbers $m_1, \dots, m_k$ are called the *exponents* (*or weights*) of the S^1-action at the point p. The exponents of an action are not canonical and their sign can be changed in pairs. Consider the sum of the exponents $S(p) = \sum_{i=1}^k m_i$. The number $S(p)$ is constant on each connected component of M^{S^1}, but may vary for different connected components.

Definition 2.1. A circle action on an oriented $2n$-dimensional manifold M will be called *even* if the sum $S(p) \equiv 0 \pmod 2$ for all $p \in M^{S^1}$, and *odd* if $S(p) \equiv 1 \pmod 2$ for all $p \in M^{S^1}$.

Lemma 2.1. *(See [5].) Let M be an oriented, connected, compact $2n$-dimensional manifold. Assume M is π_2-finite and admits a smooth S^1 action. Then $S(p_1) = S(p_2)$ for all $p_1, p_2 \in M^{S^1}$. In particular, the S^1-action is either even or odd.*

Proposition 2.1. *(See [5].) Let M be an oriented, connected, compact, π_2-finite $2n$-dimensional manifold, admitting a smooth S^1-action. Let $\mathbb{Z}_2 = \{\pm 1\}$ be the subgroup of S^1 generated by the involution. Let X be a connected component of the $\mathbb{Z}_2$-fixed point set $M^{\mathbb{Z}_2}$ such that $X \cap M^{S^1} \neq \emptyset$. Then*

$$\operatorname{codim}(X) \equiv 0 \pmod 4 \quad \textit{if the action is even,}$$
$$\operatorname{codim}(X) \equiv 2 \pmod 4 \quad \textit{if the action is odd.}$$

3. Elliptic genera on π_2-finite manifolds

The elliptic genus can be defined as

$$\Phi(M) = \operatorname{sign}\Big(M, \bigotimes_{i=1}^{\infty} \bigwedge\nolimits_{q^i} T \otimes \bigotimes_{j=1}^{\infty} S_{q^j} T\Big) = \sum_{j \geqslant 0} \operatorname{sign}(M, R_j),$$

where $T = TM \otimes \mathbb{C}$ and $\operatorname{sign}(M, E)$ denotes the index of the signature operator twisted by the bundle E, $S_a T = \sum_{j=0}^{\infty} a^j S^j T$, $\bigwedge_a T = \sum_{j=0}^{\infty} a^j \bigwedge^j T$, and $S^j T$, $\bigwedge^j T$ denote the j-th symmetric and exterior tensor powers of T, respectively [6]. The first few R_j's are, $R_0 = 1$, $R_1 = 2T$, etc. Witten conjectured the rigidity of this genus for Spin manifolds [9], which was proved by Bott and Taubes [1], and others. We proved that such a rigidity also holds on non-Spin π_2-finite manifolds [4]. Moreover, the elliptic genus $\Phi(M)$ has modular properties and by changing coordinate in q (changing cusp) one obtains a different expression. Namely,

$$\tilde{\Phi}(M) = \hat{A}\left(M, \bigotimes_{i=2j+1>0} \bigwedge\nolimits_{-q^i} T \otimes \bigotimes_{i=2j>0} S_{q^i} T\right) = \frac{1}{q^{\dim(M)/8}} \cdot \left(\sum_{j \geqslant 0} \hat{A}(M, R'_j)\right),$$

where $R'_0 = 1$, $R'_1 = -T$, etc.

Theorem 3.1. *(See [5].) Let M be an oriented, connected, compact $4n$-dimensional manifold with finite second homotopy group. Assume M admits a (non-trivial) smooth S^1-action, and let $\mathbb{Z}_2 = \{\pm 1\}$ the subgroup generated by the involution $-1 \in S^1$.*

- *If the action is odd, then $\Phi(M) = 0$ and $\tilde{\Phi}(M) = 0$.*
- *If the action is even and $\operatorname{codim}(Y) \geqslant 4r$ for all the connected components Y of $M^{\mathbb{Z}_2}$ that contain S^1-fixed points, then the characteristic numbers $\hat{A}(M, R'_j)$ vanish for $1 \leqslant j \leqslant r - 1$. If $r \geqslant n/2$ then $\Phi(M)$ does not depend on the variable q and $\Phi(M) = \operatorname{sign}(M)$. If $r > n/2$, then $\Phi(M) = 0$, $\tilde{\Phi}(M) = 0$.*

Corollary 3.1. *Let M be a connected π_2-finite manifold with smooth S^1-action. Let $\mathbb{Z}_2 = \{\pm 1\}$ be the subgroup generated by the involution $-1 \in S^1$. Assume that the induced $\mathbb{Z}_2$ action is effective. If $\hat{A}(M, T) \neq 0$ then the S^1 action is even and the fixed point manifold $M^{\mathbb{Z}_2}$ has at least one connected component of codimension 4 with non-empty intersection with the S^1-fixed point set M^{S^1}.*

4. Totally geodesic submanifolds

The assumption of positive sectional curvature imposes strong restrictions on the totally geodesic submanifolds as the classic theorem of Frankel shows [3].

Theorem 4.1. *(See [3].) Let M be a connected Riemannian manifold of positive sectional curvature. Suppose N_1 and N_2 are totally geodesic submanifolds. If $\dim(N_1) + \dim(N_2) \geqslant \dim(M)$ then $N_1 \cap N_2 \neq \emptyset$.*

Theorem 4.2. *(See [8].) Let M be a connected Riemannian manifold of positive sectional curvature. Suppose N is a connected totally geodesic submanifold of codimension k. Then the inclusion $j: N \hookrightarrow M$ is $(\dim(M) - 2k + 1)$-connected.*

Let $j_!: H^*(N, \mathbb{Z}) \to H^{*+k}(M, \mathbb{Z})$ be the push-forward in cohomology, and define $u := j_!(1) \in H^k(M, \mathbb{Z})$. By Theorem 4.2, the map $\cup u: H^i(M, \mathbb{Z}) \to H^{i+k}(M, \mathbb{Z})$ is injective for $k - 1 < i \leqslant \dim(M) - 2k + 1$ and surjective for $k - 1 \leqslant i < \dim(M) - 2k + 1$. It is not hard to check that one can replace the coefficients $\mathbb{Z}$ by $\mathbb{Z}_2$.

5. Proof of the theorem

Let M be a π_2-finite manifolds. Thanks to theorem 1.1 of [2] we can assume M is non-Spin. In order to get a contradiction, assume $\hat{A}(M, T) \neq 0$. Let $\dim(M) = 4m \geqslant 12$. Since M is even-dimensional and oriented with positive sectional curvature, it is simply connected by the classical Synge theorem.

Let $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \subset G$ a subgroup that acts effectively and isometrically on M. Since G is connected, every element of H is contained in some S^1-subgroup of G. Let $g_1, g_2, g_3 \in H$ denote the non-trivial elements. Each g_i is contained in a circle subgroup $S^1_i \subset G$ acting on M. The action of S^1_i is even and, by Corollary 3.1, there is a connected component F_i of M^{g_i} of codimension 4 containing S^1_i-fixed points. Since M has positive sectional curvature, the other components of M^{g_i} containing S^1_i-fixed points can only be isolated points, if any, by Theorem 4.1. The inclusion

$F_i \hookrightarrow M$ is $(4m-7)$-connected by Theorem 4.2. Thus $\pi_1(F_i)=\pi_1(M)=1$, $\pi_2(F_i)=\pi_2(M)$ so that F_i is also π_2-finite.

Consider $N=\bigcap_i F_i \subseteq M^H$. Notice that N is a compact subset of M^H disjoint from other components of M^H, therefore it must be a submanifold. Furthermore, since M^H is totally geodesic, then N is also totally geodesic. Let $p \in N$, the action of H on T_pM splits into one-dimensional real H-representations. It is not hard to check that the subspace invariant by the infinitesimal action of H in T_pM has codimension 6, i.e. the connected component of N containing p has codimension 6 in M, and codimension 2 in F_i, $i=1,2,3$. Since $\dim M \geqslant 12$, Theorem 4.1 implies that N is connected.

Since N is totally geodesic, consider the map $j: N \hookrightarrow F_1$, so that cup product with $u := j_!(1) \in H^2(F_1,\mathbb{Z}_2)$ gives an isomorphism $\cup u : H^i(F_1,\mathbb{Z}_2) \xrightarrow{\cong} H^{i+2}(F_1,\mathbb{Z}_2)$, for $1<i\leqslant 4m-8$. Together with the fact that $\cup u : H^1(F_1,\mathbb{Z}_2) \to H^3(F_1,\mathbb{Z}_2)$ is onto, we get that $H^{2j+1}(F_1,\mathbb{Z}_2)=0$, for $j \geqslant 0$. Therefore we have two cases: 1. $u=0$ and F_1 is a $\mathbb{Z}_2$-cohomology sphere, or 2. $u \neq 0$ and $H^*(F_1,\mathbb{Z}_2)$ is generated by $H^2(F_1,\mathbb{Z}_2)$. Case number 1 cannot occur since we are assuming that M is non-Spin and $H^2(F_1,\mathbb{Z}_2)=H^2(M,\mathbb{Z}_2)\neq 0$.

Case number 2 does not occur either. Here, the argument in [2] is applied by substituting M^H by N. As in the Spin case, if one assumes that $u \neq 0$ then

$$H^i(F_1,\mathbb{Z}_2) \cong H^i(N,\mathbb{Z}_2), \quad \text{for every } i. \tag{1}$$

On the other hand, since $4m \geqslant 12$ and $N \hookrightarrow F_1$ is $(4m-7)$-connected, we get the following,

$$H^{2j+1}(N,\mathbb{Z}_2)=0, \qquad H^2(F_1,\mathbb{Z}_2)\cong H^2(N,\mathbb{Z}_2),$$

and multiplication with $u|_N \in H^2(N,\mathbb{Z}_2)$ gives an isomorphism $H^i(N,\mathbb{Z}_2) \xrightarrow{\cong} H^{i+2}(N,\mathbb{Z}_2)$, for $1<i\leqslant 4m-10$. Thus,

$$\dim H^*(F_1,\mathbb{Z}_2)=\dim H^*(N,\mathbb{Z}_2)+\dim H^2(F_1,\mathbb{Z}_2). \tag{2}$$

Now, (2) and (1) together imply that $H^2(F_1,\mathbb{Z}_2)=0$, which is isomorphic to $H^2(M,\mathbb{Z}_2)\neq 0$.

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