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Probability Theory

Average Euler characteristic of random real algebraic varieties

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Abstract

We determine the expected curvature polynomial of real projective varieties given as the zero set of independent random polynomials with Gaussian distribution, which is invariant under the orthogonal group. In particular, the expected Euler characteristic of such random real projective varieties is found. This considerably extends previously known results on the number of roots, the volume, and the Euler characteristic of the real solution set of random polynomial equations. ***To cite this article: P. Bürgisser, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

La caractéristique d'Euler moyenne des variétés algébriques réelles aléatoires. Dans cet article, nous déterminons l'espérance du polynôme de courbure d'une variété projective réelle qui est donnée comme ensemble de zéros de polynômes aléatoires, avec une distribution gaussienne qui est invariante par le groupe orthogonal. En particulier, nous explicitons la caractéristique d'Euler de telles variétés projectives réelles aléatoires. Ces résultats généralisent considérablement la connaissance du nombre de zéros, du volume, et de la caractéristique d'Euler, des ensembles de zéro des systèmes de polynômes aléatoires. ***Pour citer cet article : P. Bürgisser, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Soit $H_{d,n}$ l'espace vectoriel des polynômes réels homogènes de degré d en les variables $X_0,\dots,X_n$. Soit $f=\sum_\alpha f_\alpha X_0^{\alpha_0}\cdots X_n^{\alpha_n}\in H_{d,n}$ avec des coefficients f_α qui sont des variables aléatoires indépendantes gaussiennes centrées avec variance $\binom{d}{\alpha}:=\frac{d!}{\alpha_0!\cdots\alpha_n!}$. On peut montrer que la distribution de probabilité induite sur $H_{d,n}$ est invariante par le groupe orthogonal $O(n+1)$. Nous dirons que f est distribuée selon Kostlan [7]. Considerons un système $f_1(x)=0,\dots,f_n(x)=0$ de tels polynômes aléatoires. Shub et Smale [14] ont montré que l'espérance du nombre de zéros dans l'espace projectif réel $\mathbb{P}^n$ est égale à $\sqrt{d_1\cdots d_n}$, c'est-à-dire la racine carrée du produit des degrés des polynômes. Dans le cas où tous les polynômes ont le même degré, ce résultat a été également établi par Kostlan [7].

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Dans le cas sous-determiné $f_1(x) = 0, \dots, f_s(x) = 0$ $(s < n)$, l'ensemble des zéros $\mathcal{Z}(f_1, \dots, f_s)$ est une sous-variété de l'espace projectif réel $\mathbb{P}^n$. Comme généralisation de la cardinalité on peut prendre le volume en dimension $n - s$ ou bien la caractéristique d'Euler. On a $\mathbb{E}\,\mathrm{vol}(\mathcal{Z}(f_1, \dots, f_s)) = \sqrt{d_1 \cdots d_s}\,\mathrm{vol}(\mathbb{P}^{n-s})$ pour un système de polyômes f_i distribué selon Kostlan [7].

Podkorytov [11] a défini le *paramètre* $\delta(f) := \frac{\mathbb{E}\|Df(x)\|^2}{n\mathbb{E}f(x)^2}$ d'un polynôme aléatoire avec une distribution gaussienne centrée, qui est invariante par le groupe orthogonal $O(n+1)$. Ici x est un point quelconque dans S^n et $Df(x): T_xS^n \to \mathbb{R}$ est la dérivée de f interprété comme fonction $f: S^n \to \mathbb{R}$ (noter que $\delta(f)$ est indépendant de x par invariance et homogénéité). Le paramètre d'un polynôme $f \in H_{d,n}$ distribué selon Kostlan est égale au degré d. En utilisant la théorie de Morse, Podkorytov [11] a montré que l'espérance de la caractéristique d'Euler de l'hypersurface $\mathcal{Z}(f) \subseteq \mathbb{P}^n$ peut être écrite comme

$$\mathbb{E}\chi\big(\mathcal{Z}(f)\big) = I_n\big(\sqrt{\delta(f)}\big)/I_n(1) \quad \text{si } n \text{ est impair},$$

où $I_n(\sqrt{\delta}) := \int_0^{\sqrt{\delta}} (1 - x^2)^{\frac{n-1}{2}}\,\mathrm{d}x$. Dans le cas où n est pair, nous remarquons que, presque sûrement, $\mathcal{Z}(f)$ est une variété différentielle compacte de dimension impaire ; alors sa caractéristique d'Euler est zéro.

Nous étendons le résultat de Podkorytov au cas d'une codimension plus grande :

Théorème 1.1. *Soient $f_1 \in H_{d_1,n}, \dots, f_s \in H_{d_s,n}$ des polyômes indépendants avec une distribution gaussienne centrée qui est invariante par le groupe orthogonal. Soit δ_i le paramètre de f_i et supposons que $s \leqslant n$ et que $n - s$ est pair. Alors $\mathbb{E}\chi(\mathcal{Z}(f_1, \dots, f_s)) = a_0 + a_2 + \cdots + a_{\frac{n-s}{2}}$, où les a_j sont les coefficients de la série de puissances suivante :*

$$\sum_{j=0}^{\infty} a_{2j} T^{2j} := \prod_{i=1}^{s} \frac{\delta_i^{1/2}}{(1 - (1 - \delta_i)T^2)^{1/2}}.$$

Dans le cas où tous les paramètres sont égaux à δ, nous avons

$$\mathbb{E}\chi\big(\mathcal{Z}(f_1, \dots, f_s)\big) = \delta^{s/2} \sum_{k=0}^{\frac{n-s}{2}} C_k^{(s)} (1 - \delta)^k = (-1)^{\frac{n-s}{2}}\, C_{\frac{n-s}{2}}^{(s)}\, \delta^{\frac{n}{2}} + O(\delta^{\frac{n}{2}-1}) \quad (\delta \to \infty)$$

où $C_0^{(s)} = 1$ et $C_k^{(s)} = \frac{s(s+2)(s+4)\cdots(s+2k-2)}{k!2^k}$ si $k > 0$.

La preuve repose sur l'étude des coefficients de courbure de la variété différentielle formée par les solutions et sur la formule principale de la géometrie intégrale [5,4,6]. En fait, nous déterminons l'espérance du polynôme de courbure (Éq. (5)) qui contient la caractéristique d'Euler.

1. Introduction

Let $H_{d,n}$ denote the vector space of homogeneous real polynomials of degree d in the variables $X_0, \dots, X_n$. Let $f = \sum_\alpha f_\alpha X_0^{\alpha_0} \cdots X_n^{\alpha_n} \in H_{d,n}$ be such that the coefficients f_α are independent centered Gaussian random variables with variance $\binom{d}{\alpha} := \frac{d!}{\alpha_0! \cdots \alpha_n!}$. The induced probability distribution on $H_{d,n}$ can be shown to be invariant under the natural action of the orthogonal group $O(n+1)$. We will say that such f is a *Kostlan distributed* random polynomial [7]. Consider a system $f_1(x) = 0, \dots, f_n(x) = 0$ of such random polynomials. Shub and Smale [14] showed that its expected number of real roots in real projective space $\mathbb{P}^n$ equals $\sqrt{d_1 \cdots d_n}$, i.e., the square root of the product of the degrees d_i of the polynomials. This result was also found by Kostlan [7] in the case where all polynomials have the same degree. Extensions of this result can be found in [12,9,8]. A different proof of the Kostlan–Shub–Smale theorem based on the Rice formula from the theory of random fields was recently given by Azaïs and Wschebor [2]. Wschebor [17], for the first time, analyzed the variance of the number of real roots.

Considerably less is known for the underdetermined case $f_1(x) = 0, \dots, f_s(x) = 0$ $(s < n)$ where the zero set $\mathcal{Z}(f_1, \dots, f_s)$ is an algebraic subvariety of $\mathbb{P}^n$. As a measure of its size different choices come to mind: one possible choice is the $(n-s)$-dimensional volume. Another generalization of cardinality to higher dimensional solutions sets is the Euler characteristic. Both of these measures have been considered already. For a system of Kostlan distributed f_i, we have $\mathbb{E}\,\mathrm{vol}(\mathcal{Z}(f_1, \dots, f_s)) = \sqrt{d_1 \cdots d_s}\,\mathrm{vol}(\mathbb{P}^{n-s})$, as was first observed in [7] for the case $d_1 = \cdots = d_s$.

Podkorytov [11] more generally investigated invariant centered Gaussian random polynomials $f \in H_{d,n}$ (invariant meaning that the distribution of f is invariant under the action of the orthogonal group $O(n+1)$). He defined its *parameter*

$$\delta(f) := \frac{\mathbb{E}\,\|Df(x)\|^2}{n\,\mathbb{E}\,f(x)^2},$$

where x denotes any point in S^n and $Df(x): T_xS^n \to \mathbb{R}$ is the derivative of f interpreted as a function $f: S^n \to \mathbb{R}$ (we note that $\delta(f)$ is independent of x by invariance and homogeneity). The parameter of a Kostlan distributed $f \in H_{d,n}$ equals the degree d.

Using Morse theory, Podkorytov proved in [11] that the expected Euler characteristic of the hypersurface $\mathcal{Z}(f)$ in $\mathbb{P}^n$ can be expressed by its parameter as follows:

$$\mathbb{E}\chi\big(\mathcal{Z}(f)\big) = I_n\big(\sqrt{\delta(f)}\big)/I_n(1) \quad \text{for odd } n, \tag{1}$$

where $I_n(\sqrt{\delta}) := \int_0^{\sqrt{\delta}}(1-x^2)^{\frac{n-1}{2}}\,\mathrm{d}x$. Note that if n is even, $\mathcal{Z}(f)$ is almost surely a compact odd-dimensional manifold and therefore its Euler characteristic vanishes.

We extend Podkorytov's result to higher codimension:

Theorem 1.1. *Suppose that $f_1 \in H_{d_1,n}, \ldots, f_s \in H_{d_s,n}$ are independent invariant centered Gaussian random polynomials and let δ_i denote the parameter of f_i. Suppose that $s \leqslant n$ and $n-s$ is even. Then $\mathbb{E}\,\chi(\mathcal{Z}(f_1,\ldots,f_s)) = a_0 + a_2 + \cdots + a_{\frac{n-s}{2}}$, where the a_j are the coefficients of the following power series*

$$\sum_{j=0}^{\infty} a_{2j}T^{2j} := \prod_{i=1}^{s} \frac{\delta_i^{1/2}}{(1-(1-\delta_i)T^2)^{1/2}}.$$

In the case where all parameters equal δ, we more explicitly have

$$\mathbb{E}\chi\big(\mathcal{Z}(f_1,\ldots,f_s)\big) = \delta^{s/2}\sum_{k=0}^{\frac{n-s}{2}} C_k^{(s)}(1-\delta)^k = (-1)^{\frac{n-s}{2}} C_{\frac{n-s}{2}}^{(s)}\delta^{\frac{n}{2}} + O(\delta^{\frac{n}{2}-1}) \quad (\delta \to \infty),$$

where $C_0^{(s)} = 1$ and $C_k^{(s)} = \frac{s(s+2)(s+4)\cdots(s+2k-2)}{k!\,2^k}$ for $k > 0$.

For proving this result, a direct use of Morse theory as in [11] does not seem feasible. Instead, it turned out to be essential to study the notions of curvature coefficients of the solution manifold and to employ the kinematic formula of integral geometry [5,4,6]. In fact, we determine the expectation of the so-called curvature polynomial (Eq. (5)), from which the Euler characteristic can be easily read off.

A different proof of Theorem 1.1, based on Weyl's tube formula and the Rice formula from the theory of random fields, was given in [3]. We finally remark that our work has connections with recent investigations on the geometry of random fields, cf. Adler and Taylor [15].

2. Background from differential and integral geometry

In a seminal work, Weyl [16] derived a formula for the volume of the tube $T(M,\alpha) := \{y \in \mathbb{P}^n \mid \mathrm{dist}(y,M) \leqslant \alpha\}$ of radius α around an m-dimensional compact smooth submanifold M of $\mathbb{P}^n$ (actually Weyl considered the sphere S^n). Let $s := n-m$ denote the codimension of M. Weyl [16] proved that, for sufficiently small $\alpha > 0$, $\mathrm{vol}(T(M,\alpha))$ can be written as a linear combination

$$\mathrm{vol}\big(T(M,\alpha)\big) = \sum_{0 \leqslant e \leqslant m,\, e \text{ even}} K_{s+e}(M)\,J_{n,s+e}(\alpha)/2$$

of the functions $J_{n,k}(\alpha) := \int_0^{\alpha}(\sin\rho)^{k-1}(\cos\rho)^{n-k}\,\mathrm{d}\rho$. The coefficients $K_{s+e}(M)$ depend on the curvature of M in $\mathbb{P}^n$ and will be thus called its *curvature coefficients*. Let $O_{n-1} := 2\pi^{n/2}/\Gamma(n/2)$ denote the dimensional volume of the unit sphere S^{n-1}. Following Nijenhuis [10], we define the *curvature polynomial* of M by setting

$$\mu(M;T) := \sum_{0 \leqslant e \leqslant m,\, e \text{ even}} \mu_e(M)T^e := \sum_{0 \leqslant e \leqslant m,\, e \text{ even}} \frac{K_{s+e}(M)}{O_{m-e}O_{s+e-1}}T^e.$$

For example, a linear subspace $\mathbb{P}^m$ of $\mathbb{P}^n$ satisfies $\mu(\mathbb{P}^m;T)=1$. The constant term of the curvature polynomial describes the volume of M, namely $\mu_0(M)=\operatorname{vol}(M)/\operatorname{vol}(\mathbb{P}^m)$.

The following statement can be deduced from the generalized Gauss–Bonnet theorem [1]:

Theorem 2.1. *Let M be a compact smooth submanifold of $\mathbb{P}^n$ of even dimension. Then the Euler characteristic of M can be expressed as $\chi(M)=\mu(M;1)$.*

Nijenhuis [10] gave an elegant formulation of the principal kinematic formula of integral geometry [5,4] in terms of reduced polynomial multiplication of the curvature polynomials. He formulated this for Euclidean space, but it holds in much higher generality, cf. [13,6].

Theorem 2.2. *Let M and N be compact smooth submanifolds of $\mathbb{P}^n$ of the dimensions m and p, respectively, such that $m+p\geqslant n$. Then we have*

$$\int \mu(M\cap gN;T)\,\mathrm{d}g \equiv \mu(M;T)\mu(N;T) \bmod T^{m+p-n+1},$$

where the integration is with respect to the Haar measure on the orthogonal group $O(n+1)$ scaled such that the volume of $O(n+1)$ equals 1. In particular, we have

$$\int \mu(M\cap g\mathbb{P}^p;T)\,\mathrm{d}g \equiv \mu(M;T) \bmod T^{m+p-n+1}.$$

3. Proof of Theorem 1.1

Podkorytov's result (1) takes a simpler form when expressed in terms of generating functions. We define $\chi_k(\delta):=\mathbb{E}\,\chi(\mathcal{Z}(f))$, where $f\in H_{d,2k+1}$ is an invariant centered Gaussian random polynomial with parameter δ.

We have

$$I_{2k+1}(1)=\int_0^1(1-x^2)^k\,\mathrm{d}x=\frac{2^k k!}{1\cdot 3\cdot 5\cdots(2k+1)}.$$

The binomial series yields

$$(1-Y)^{-3/2}=\sum_{k=0}^{\infty}\frac{1}{I_{2k+1}(1)}Y^k.$$

By a manipulation of formal power series we obtain from (1)

$$\begin{aligned}\sum_{k=0}^{\infty}\chi_k(\delta)\,T^{2k}&=\sum_{k=0}^{\infty}\frac{I_{2k+1}(\delta^{1/2})}{I_{2k+1}(1)}\,T^{2k}=\sum_{k=0}^{\infty}\frac{1}{I_{2k+1}(1)}\int_0^{\sqrt{\delta}}(1-x^2)^k\,\mathrm{d}x\,T^{2k}\\&=\int_0^{\sqrt{\delta}}\sum_{k=0}^{\infty}\frac{\big((1-x^2)T^2\big)^k}{I_{2k+1}(1)}\,\mathrm{d}x=\int_0^{\sqrt{\delta}}\frac{\mathrm{d}x}{(1-T^2+x^2T^2)^{3/2}}.\end{aligned}$$

Using

$$\int_0^{\xi}\frac{\mathrm{d}x}{(A+Bx^2)^{3/2}}=\frac{\xi}{A\sqrt{A+B\xi^2}}$$

we arrive at

$$\chi(\delta;T):=\sum_{k=0}^{\infty}\chi_k(\delta)\,T^{2k}=\frac{\delta^{1/2}}{(1-T^2)(1-(1-\delta)T^2)^{1/2}}. \tag{2}$$

We connect now the expected Euler characteristic to expected curvature coefficients:

Lemma 3.1. *Suppose that $f \in H_{d,n}$ has $O(n+1)$-invariant distribution with parameter δ. Then we have $\mathbb{E}\mu_0(\mathcal{Z}(f)) = \chi_0(\delta)$ and $\mathbb{E}\mu_{2k}(\mathcal{Z}(f)) = \chi_k(\delta) - \chi_{k-1}(\delta)$ for $1 \leqslant k \leqslant (n-1)/2$. In particular, $\mathbb{E}\mu_{2k}(\mathcal{Z}(f))$ depends only on k and δ and not on the dimension n of the ambient space.*

Proof. We denote by f' and f'' the restrictions of f to $\mathbb{R}^{2k+2}$ and $\mathbb{R}^{2k}$, respectively. Theorem 2.1 implies that

$$\chi_k(\delta) = \sum_{i=0}^{k} \mathbb{E}\mu_{2i}\big(\mathcal{Z}(f')\big), \qquad \chi_{k-1}(\delta) = \sum_{i=0}^{k-1} \mathbb{E}\mu_{2i}\big(\mathcal{Z}(f'')\big). \tag{3}$$

We use the following result that is easily deduced from Theorem 2.2 by taking $N = \mathbb{P}^{n'}$:

Lemma 3.2. *Suppose $f \in H_{d,n}$ has $O(n+1)$-invariant distribution and $n' \leqslant n$. Then the restriction $f' \in H_{d,n'}$ of f to $\mathbb{R}^{n'+1}$ has $O(n'+1)$-invariant distribution and has the same parameter. For $0 \leqslant e < n'$, e even, we have $\mathbb{E}\mu_e(\mathcal{Z}(f')) = \mathbb{E}\mu_e(\mathcal{Z}(f))$.*

This lemma implies $\mathbb{E}\mu_{2i}(\mathcal{Z}(f)) = \mathbb{E}\mu_{2i}(\mathcal{Z}(f')) = \mathbb{E}\mu_{2i}(\mathcal{Z}(f''))$ for $0 \leqslant i \leqslant k-1$ and $\mathbb{E}\mu_{2k}(\mathcal{Z}(f)) = \mathbb{E}\mu_{2k}(\mathcal{Z}(f'))$. Subtracting the two equations (3), we obtain $\chi_k(\delta) - \chi_{k-1}(\delta) = \mathbb{E}\mu_{2k}(\mathcal{Z}(f))$ as claimed. The assertion for $k = 0$ is obvious. □

By Lemma 3.1 the formal power series

$$\mu(\delta; T) := \sum_{k=0}^{\infty} \mathbb{E}\mu_{2k}\big(\mathcal{Z}(f)\big) T^{2k}$$

satisfies $\mu(\delta; T) = (1 - T^2)\chi(\delta; T)$. Eq. (2) implies that

$$\mathbb{E}\mu(\delta; T) = \delta^{1/2}\big(1 - (1-\delta)T^2\big)^{-1/2}. \tag{4}$$

Suppose now that $f_i \in H_{d_i,n}$ are as in Theorem 1.1. Write $\underline{f} := (f_1, \ldots, f_{s-1})$. Theorem 2.2 tells us that for almost all $\underline{f}$, f_s

$$\int \mu\big(\mathcal{Z}(\underline{f}) \cap g\mathcal{Z}(f_s); T\big)\, \mathrm{d}g \equiv \mu\big(\mathcal{Z}(\underline{f}); T\big)\mu\big(\mathcal{Z}(f_s); T\big) \bmod T^{n-s+1},$$

where the integral is over $O(n+1)$ with respect to the Haar measure scaled to 1. We take the expectation with respect to $\underline{f}$ and f_s. Taking their independence into account we get

$$\int \mathbb{E}_{\underline{f}, f_s} \mu\big(\mathcal{Z}(\underline{f}) \cap g\mathcal{Z}(f_s); T\big)\, \mathrm{d}g \equiv \mathbb{E}\mu\big(\mathcal{Z}(\underline{f}); T\big)\mathbb{E}\mu\big(\mathcal{Z}(f_s); T\big) \bmod T^{n-s+1}.$$

By invariance, the integrand does not depend on g and equals $\mathbb{E}\mu(\mathcal{Z}(f_1, \ldots, f_s); T)$. We conclude by induction that

$$\mathbb{E}\mu\big(\mathcal{Z}(f_1, \ldots, f_s); T\big) \equiv \mathbb{E}\mu\big(\mathcal{Z}(f_1); T\big) \cdots \mathbb{E}\mu\big(\mathcal{Z}(f_s); T\big) \bmod T^{n-s+1}.$$

Plugging in the explicit expression (4) for $\mathbb{E}\mu(\mathcal{Z}(f_i); T)$ we get

$$\mathbb{E}\mu\big(\mathcal{Z}(f_1, \ldots, f_s); T\big) \equiv \prod_{i=1}^{s} \frac{\delta_i^{1/2}}{(1 - (1-\delta_i)T^2)^{1/2}} \bmod T^{n-s+1}. \tag{5}$$

The claim about the Euler characteristic follows now by Theorem 2.1. □

Eq. (5) is of independent interest and can be seen as the main result of this work. By evaluating this equation at $T = 0$ we get

$$\mathbb{E}\mu_0\big(\mathcal{Z}(f_1, \ldots, f_s)\big) = \mathbb{E}\,\mathrm{vol}\big(\mathcal{Z}(f_1, \ldots, f_s)\big) / \mathrm{vol}(\mathbb{P}^{n-s}) = (\delta_1 \cdots \delta_s)^{1/2}.$$

We note that this is a generalization of the results by Kostlan [7] and Shub and Smale [14] since this holds for any invariant centered Gaussian random polynomials.

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