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C. R. Acad. Sci. Paris, Ser. I 345 (2007) 593–598

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Numerical Analysis

Stabilized methods for stiff stochastic systems

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Received 12 July 2007; accepted after revision 28 September 2007

Available online 7 November 2007

Presented by Philippe G. Ciarlet

Abstract

Stiff stochastic systems are usually solved numerically by (semi-)implicit methods, since explicit methods, such as the Euler–Maruyama scheme, face severe stepsize reductions. This comes at the cost of solving linear algebra systems at each step and can be expensive for large systems and complicated to implement for complex problems. In this Note, we introduce a new class of explicit methods for stochastic differential equations with multi-dimensional Wiener processes, with much better stability properties (in the mean square sense) than existing explicit methods. These new methods are as easy to implement as standard explicit schemes but much more efficient for handling stiff stochastic problems. ***To cite this article: A. Abdulle, S. Cirilli, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Schémas explicites stabilisés pour la résolution d'équations différentielles stochastiques raides. Les équations différentielles stochastiques (EDS) raides sont usuellements résolues par des schémas (semi-)implicites, car l'utilisation de schémas explicites, comme par exemple celui d'Euler–Maruyama, entraîne une sévère réduction du pas de temps. L'utilisation de schémas implicites implique la résolution de systèmes linéaires à chaque pas de temps. Cette procédure est coûteuse pour des grands systèmes d'équations et peut être difficile à réaliser pour des systèmes complexes. Dans cette Note, nous proposons une nouvelle classe de schémas explicites pour des EDS avec un processus de Wiener multidimensionel, qui ont des propriétés de stabilité (en moyenne quadratique) bien plus favorables que les schémas explicites existants. Ces nouveaux schémas sont aussi aisés à réaliser que les schémas explicites traditionnels, mais plus efficaces pour résoudre numériquement des équations différentielles stochastiques raides. ***Pour citer cet article : A. Abdulle, S. Cirilli, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

La résolution d'équations différentielles ordinaires (EDO) raides par des schémas d'intégration explicites avec des domaines de stabilité étendus a été introduite par Yan Chzao Din (1958) et Guillou et Lago (1960) [5]. Ces schémas ont été ensuite développés par de nombreux auteurs [1,2,8,13]. La bonne performance de ces méthodes est due à la croissance quadratique, avec le nombre d'étages, du domaine de stabilité le long de l'axe réel négatif. L'avantage

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.10.009

de telles méthodes est d'éviter l'emploi d'algèbre linéaire, nécessaire lorsque des schémas implicites sont utilisés. Remarquons que des schémas explicites traditionnels ne peuvent être employés pour des problèmes raides qu'avec des pas de temps extrêmement petits. Dans cette Note, nous proposons une nouvelle classe de schémas explicites pour des EDS raides avec un processus de Wiener multidimensionel. Nous introduisons une classe de schémas d'intégration explicites avec des propriétés favorables de stabilité en moyenne quadratique. Une comparaison avec des schémas explicites existants montre que ces nouvelles méthodes sont plus efficaces pour la résolution numérique d'EDS raides.

1. Introduction

Stiff stochastic problems arise in the modeling of many biological, chemical, physical and economical systems. Stiffness is concerned with (local) properties of a differential equation which can affect the stability of a numerical method [7,5]. The stability concept under consideration is mean-square stability. In this Note we consider the following stochastic differential equations (SDEs) written in Stratonovich form as[1]

$$\mathrm{d}Y = f(t,Y)\,\mathrm{d}t + \sum_{l=1}^{M} g_l(t,Y)\circ \mathrm{d}W_l(t), \qquad Y(t_0)=Y_0, \tag{1}$$

where $Y(t)$ is a random variable with value in $\mathbb{R}^d$, $f:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ is called the drift function, $g:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ is called the diffusion function and $W_l(t)$ are independent Wiener processes. We will assume usual conditions on f and g (continuity, uniform Lipschitz continuity with respect to the second variable, linear growth condition) and on Y_0 (independence of the Wiener processes and finite second order moment) to ensure existence and uniqueness of a (mean square bounded) strong solution of (1) (see for example [10, 5.2] for details). In what follows, we assume for simplicity that (1) is given in autonomous form. We consider the following class of Runge–Kutta methods for the numerical solution of SDEs proposed in [12]:

$$\bar{Y}_i = Y_n + h\sum_{j=1}^{i-1} a_{ij} f(\bar{Y}_j) + \sum_{j=1}^{i-1}\alpha_{ij}\sum_{l=1}^{M} J_{n_l} g_l(\bar{Y}_j), \qquad i=1,\dots,m,$$

$$Y_{n+1} = Y_n + h\sum_{i=1}^{m} b_i f(\bar{Y}_i) + \sum_{i=1}^{m}\beta_i\sum_{l=1}^{M} J_{n_l} g_l(\bar{Y}_i) \tag{2}$$

where $J_{n_l} = W_l(t_{n+1}) - W_l(t_n)$ is a Wiener increment drawn from a normal distribution with zero mean and variance $h = t_{n+1} - t_n$. Stability analysis for numerical methods is motivated by the question of the choice of the stepsize h for (2) in order to reproduce the characteristic dynamics of the true solution. In order to investigate such a question for numerical schemes we consider a linear scalar test problem simple enough to allow analysis [6]

$$\mathrm{d}Y = \lambda Y\,\mathrm{d}t + \mu Y\circ \mathrm{d}W(t), \qquad Y(t_0)=Y_0, \tag{3}$$

where $\lambda,\mu\in\mathbb{C}$. The solution of (3), $Y(t)=Y_0\exp(\lambda t+\mu W(t))$, is mean-square stable if and only if

$$\lim_{t\to\infty}\mathbb{E}\big(|Y(t)|^2\big)=0 \iff (\lambda,\mu)\in\mathcal{S}_{\mathrm{SDE}} := \big\{(\lambda,\mu)\in\mathbb{C}^2;\ \Re\lambda+\Re\mu^2<0\big\}, \tag{4}$$

where the right-hand side of (4) will be referred as the stability domain of the test equation (3). For scalar nonlinear problems (1) the test equation (3) gives insight into the behavior of a numerical method by linearization around fixed points. Applying the numerical scheme (2) to (3), squaring the results, and taking the expectations, we obtain

$$\mathbb{E}\big(|Y_{n+1}|^2\big) = R(p,q)\,\mathbb{E}\big(|Y_n|^2\big),$$

where $p=h\lambda$, $q=\sqrt{h}\mu$ and where $R(p,q)$ is a polynomial in $\Re p, \Im p, \Re q, \Im q$. The numerical method is mean-square stable for this test problem if and only

$$\lim_{n\to\infty}\mathbb{E}\big(|Y_n|^2\big)=0 \iff (h\lambda,\sqrt{h}\mu)\in\mathcal{S} := \big\{p,q\in\mathbb{C};\ R(p,q)<1\big\}. \tag{5}$$

[1] The Stratonovich integral is chosen here for its 'ordinary chain rule' property. Recall that any Itô stochastic differential equation can be converted into Stratonovich form. Thus, the numerical methods proposed in this paper apply to general SDEs.

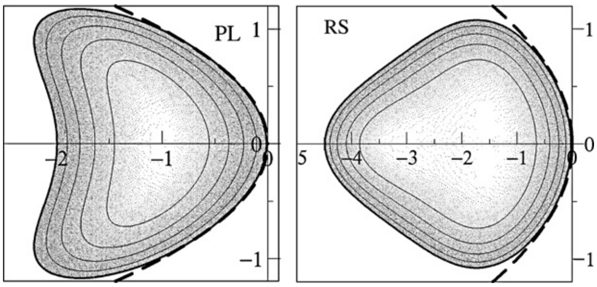

Fig. 1. Mean-square stability domains for the PL method (left picture) and the RS method (right picture).

Fig. 1. Domaine de stabilité en moyenne quadratique pour le schéma PL (à gauche) et le schéma RS (à droite).

In the following, we restrict ourselves to parameters $\lambda, \mu \in \mathbb{R}$. In Fig. 1 we plotted the stability domain of two explicit numerical methods in the $(p,q) = (h\lambda, \sqrt{h}\mu)$ plan. PL stands for the Platen method [7] and RS is a method proposed in [4]. Both methods belong to the family of methods (2) and have the same (strong or weak) order of convergence as the S-ROCK methods we will introduce below. The left part of the parabola with a boundary given by a dotted curve represents the stability region of the test equation (3). We notice that some authors use the scaling $(p,q) = (h\lambda, h\mu^2)$ for stability region diagrams for which the stability region of the test problem becomes a wedge (see [6]). We see that if $(\lambda, \mu) \in \mathcal{S}$ with $|\lambda|, |\mu| \gg 1$ (stiffness), severe stepsize reductions occur for PL or RS methods to be stable This property is shared by all other classical explicit stochastic methods [7]. To quantify stability domains of numerical methods, we start by defining a 'portion' of the true stability domain (4) by

$$\mathcal{S}_{\mathrm{SDE},r} = \left\{(p,q) \in [-r, 0] \times \mathbb{R};\ |q| \leqslant \sqrt{-p}\right\}, \tag{6}$$

where $r > 0$. We next consider for the numerical stability domain $\mathcal{S}$ two parameters l and d defined by

$$l = \max\left\{|p|;\ p < 0,\ [p, 0] \in \mathcal{S}\right\}, \quad d = \max\left\{r > 0;\ \mathcal{S}_{\mathrm{SDE},r} \in \mathcal{S}\right\}. \tag{7}$$

It is clear that $d \leqslant l$. We notice that l is a parameter of the stability domain which corresponds to the noise-free (i.e. ODE) behavior. For SDEs, large d is thus the relevant quantity to maximize. For the above methods, we have $l_{\mathrm{PL}} = 2$, $l_{\mathrm{RS}} = 4.5$ and $d_{\mathrm{PL}} = 1$, $d_{\mathrm{RS}} \simeq 0.56$.

2. The S-ROCK methods

In what follows we construct a class of numerical methods with extended mean square stability properties. The idea is to extend a class of explicit stabilized numerical methods for dissipative stiff problems introduced by Yan Chzao Din (1958) and Guillou et Lago (1960) [5, Sect. IV.2] and further developed in [1,2,8,13]. Such methods have proved to be very efficient for large stiff deterministic systems. Their efficiency is due to stability domains increasing *quadratically* with the stage number of the numerical method with stability functions given by shifted Chebyshev-like polynomials. Unlike standard methods, they rely on:

(i) implementation of a *family* of numerical schemes indexed by the stage number;
(ii) adaptive choice of a particular member of the family (at each step) to perform a stable integration given the desired timestep h and the (local) stiffness property of the problem.

We define the m-stage stochastic orthogonal Runge–Kutta Chebyshev method (S-ROCK) by

$$K_0 = Y_n, \quad K_1 = Y_n + h\frac{\omega_1}{\omega_0} f(K_0),$$

$$K_j = 2h\omega_1 \frac{T_{j-1}(\omega_0)}{T_j(\omega_0)} f(K_{j-1}) + 2\omega_0 \frac{T_{j-1}(\omega_0)}{T_j(\omega_0)} K_{j-1} - \frac{T_{j-2}(\omega_0)}{T_j(\omega_0)} K_{j-2}, \quad j = 2, \ldots, m-2,$$

$$K_{m-1} = 2h\omega_1 \frac{T_{m-2}(\omega_0)}{T_{m-1}(\omega_0)} f(K_{m-2}) + 2\omega_0 \frac{T_{m-2}(\omega_0)}{T_{m-1}(\omega_0)} K_{m-2} - \frac{T_{m-3}(\omega_0)}{T_{m-1}(\omega_0)} K_{m-3} + \sum_{l=1}^{M} \alpha J_{n_l} g_l(K_{m-2}),$$

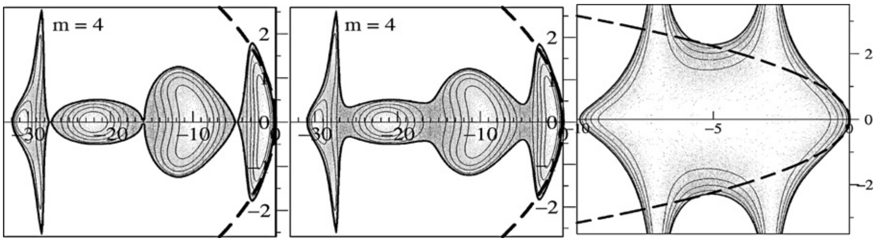

Fig. 2. Mean-square stability domains for the S-ROCK methods with $m = 4$ stages and $\eta = 0, 0.05, 11.0$.

Fig. 2. Stabilité en moyenne quadratique des schémas S-ROCK à $m = 4$ étages pour $\eta = 0, 0.05, 11.0$.

$$Y_{n+1} = K_m = 2h\omega_1 \frac{T_{m-1}(\omega_0)}{T_m(\omega_0)} f(K_{m-1}) + 2\omega_0 \frac{T_{m-1}(\omega_0)}{T_m(\omega_0)} K_{m-1} - \frac{T_{m-2}(\omega_0)}{T_m(\omega_0)} K_{m-2} + \sum_{l=1}^{M} J_{n_l}\big(\beta g_l(K_{m-2}) + \gamma g_l(K_{m-1})\big), \tag{8}$$

where $\omega_0 = 1 + \frac{\eta}{m^2}$, $\omega_1 = \frac{T_m(\omega_0)}{T'_m(\omega_0)}$ and $T_j(z)$ is the Chebyshev polynomial of degree j. These polynomials satisfy a recurrence relation which has been used in the definition of the S-ROCK methods and depend on a 'damping' parameter $\eta \in [0, \infty)$. By varying this parameter we can change the shape of the stability domain of the corresponding method (see Fig. 2). The convergence properties of the S-ROCK methods for SDEs with multidimensional Wiener processes is discussed below.

Theorem 2.1. *For $m \geqslant 2$, the methods* (8) *satisfy*

$$\mathbb{E}\big(\big|Y_N - Y(\tau)\big|\big) \leqslant Ch^{1/2}, \qquad \big|\mathbb{E}\big(p(Y_N)\big) - \mathbb{E}\big(p\big(Y(\tau)\big)\big)\big| \leqslant Ch \tag{9}$$

for any fixed $\tau = Nh \in [0, T]$ and h sufficiently small and for all functions $p : \mathbb{R}^d \to \mathbb{R}$, 4 times continuously differentiable and for which all partial derivatives have polynomials growth, if and only if

$$2\omega_0 \alpha \frac{T_{m-1}(\omega_0)}{T_m(\omega_0)} + \beta + \gamma = 1 \quad \text{and } \alpha\gamma = \frac{1}{2}. \tag{10}$$

Remark 1. The first measure of the error in (9) is called strong convergence, the second weak convergence.

Proof. A Taylor expansion of the scheme (8) using recurrence relations for Chebyshev polynomials, gives

$$Y_{n+1} = Y_n + hf(Y_n) + \left(2\omega_0 \alpha \frac{T_{m-1}(\omega_0)}{T_m(\omega_0)} + \beta + \gamma\right) \sum_{l=1}^{M} g_l(Y_n) J_{n_l} + \alpha\gamma \sum_{l,r=1}^{M} g'_l(Y_n) g_r(Y_n) J_{n_l} J_{n_r} + \mathcal{O}(hJ_{n_l}). \tag{11}$$

Using a theorem of Milstein (see [9, Chap. 1.1]) relating local and global strong convergences, the conditions (10) and comparing this expansion with the Taylor series for (1) we obtain the strong convergence result. An analogue theorem relating local and global weak convergences has been proved by Milstein (see [9, Chap. 2.2]). The order conditions on the coefficients for schemes as (2) to have a local weak order $\mathcal{O}(h^2)$ reads $\sum_{i=1}^m b_i = (\sum_{i=1}^m \beta_i)^2 = 1$, $\sum_{i=1}^m \sum_{j=1}^{i-1} \beta_i \alpha_{ij} = 1/2$ (see [11, Chap. 5]). Expanding (2) in Taylor series one gets

$$Y_{n+1} = Y_n + h\sum_{i=1}^{m} b_i f(Y_n) + \sum_{i=1}^{m} \beta_i \sum_{l=1}^{M} J_{n_l} g_l(Y_n) + \sum_{i=1}^{m}\sum_{j=1}^{i-1} \beta_i \alpha_{ij} \sum_{l,r=1}^{M} g'_l(Y_n) g_r(Y_n) J_{n_l} J_{n_r} + \mathcal{O}(hJ_{n_l}). \tag{12}$$

Comparing (11) with (12) using again (10) shows that the above order conditions are satisfied. This prove the weak first order global convergence of the methods (8). $\square$

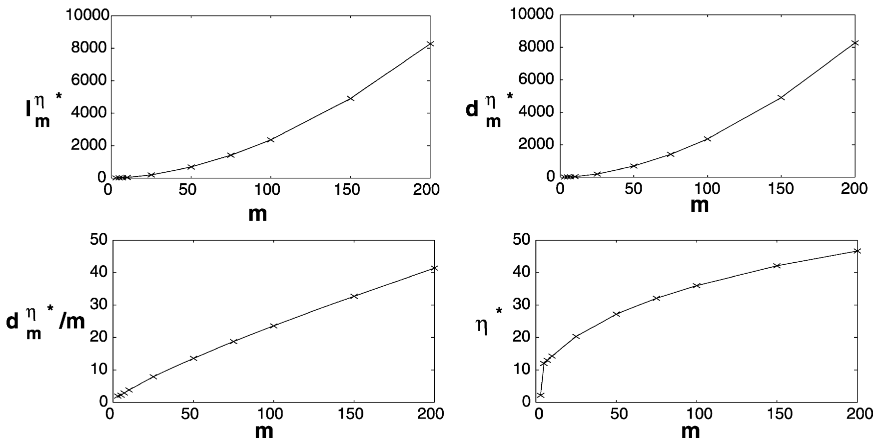

Fig. 3. Optimal values of l_m^η (upper left picture) and d_m^η (upper right picture), work/stability ratio as a function of m (lower left picture) and optimal values of η (lower right picture).

Fig. 3. Valeurs optimales de l_m^η (en haut à gauche) et de d_m^η (en haut à droite), rapport travail/stabilité comme fonction de m (en bas à gauche) et valeurs optimales de η (en bas à droite).

Theorem 2.2. *If the commutativity condition* $g_l'(t,Y)g_r(t,Y) - g_r'(t,Y)g_l(t,Y) = 0\ \forall l, r = 1,\dots,M$ *holds for the problem* (1) *or if* $M = 1$, *then the strong error* $\mathbb{E}(|Y_N - Y(\tau)|) \leqslant Ch$ *holds for the method* (8).

Proof. If the above commutativity condition holds, then we can use the relation between the multi-dimensional stochastic integrals $J_{lr} + J_{rl} = J_l J_r$, where $J_{lr} = \int_{t_0}^{t_1}\int_{t_0}^{s} \circ\, \mathrm{d}W_l(s) \circ \mathrm{d}W_r(s_1)$ and the last term in equation (11), involving $g_l'(Y_n)g_r(Y_n)$ matches with the Taylor series for (1). The strong local error is then $\mathcal{O}(hJ_{n_l})$ and applying Milsteins theorem [9, Chap. 1.1] gives the result. For $M = 1$ the argument is similar and is discussed in details in [3]. □

2.1. Stability studies

We study now the mean-square stability of the S-ROCK methods. Choosing the set of parameters $\alpha = \frac{1}{2w_0}\frac{T_m(\omega_0)}{T_{m-1}(\omega_0)}$, $\beta = -\frac{1}{2\alpha}$, $\gamma = \frac{1}{2\alpha}$, and applying the above method to the linear test problem (3), we obtain $K_j = T_j(\omega_0 + \omega_1 p)$, $j = 2,\dots,m-2$, for the internal stages and a mean-square stability function given by

$$R_m(p,q) = \frac{T_m^2(\omega_0 + \omega_1 p)}{T_m^2(\omega_0)} + Q_{2m-2,4}(p,q), \tag{13}$$

where $T_m^2(\omega_0 + \omega_1 p)$ is a shifted Chebyshev polynomial of degree $2m$ and $Q_{2m-2,4}(p,q)$ is a polynomial of degree $2m-2$ in p and of degree 4 in q satisfying $Q_{2m-2,4}(p,0) = 0$. By varying the value of η we can change the shape of the stability domain (see Fig. 2) and the values of l_m^η and d_m^η (depending now on η and m) defined in (7).

The following lemmas proved in [3] characterize the stability domains of our methods:

Lemma 2.3. *Let* $\eta \geqslant 0$. *For all m, the m-stage numerical method* (8) *has a mean square stability region* $\mathcal{S}_m^\eta$ *with* $l_m^\eta \geqslant c(\eta)m^2$, *where* $c(\eta)$ *depends only on* η.

Lemma 2.4.

$$l_m^\eta \to 2m \quad \text{for } \eta \to \infty. \tag{14}$$

The first lemma says that for η fixed, the stability domain along the p axis increases quadratically. The second lemma states that for infinite damping, the maximal portion $S_{\mathrm{SDE},d}$ which can be included in the stability domain of the S-ROCK methods grows at most linearly with the number of stage since $d_m^\eta \leqslant l_m^\eta$. The task is now for a given m by varying η to find the optimal value of d_m^η defined by $d_m^{\eta^*} = \max\{d_m^\eta;\ \eta \in [0,\infty)\}$. This is done numerically for $m \leqslant 200$. In Fig. 3 (upper plots) the optimal values $d_m^{\eta^*}$ as well as the corresponding value $l_m^{\eta^*}$ (maximum stability on the p axis) are plotted against m, the number of stages. We see that for these optimal values of η, $d_m^{\eta^*} \simeq l_m^{\eta^*}$. In the third picture of Fig. 3 (lower left plot) we study the efficiency of the method. Since for the S-ROCK methods we can use a large number of stages, these methods will be efficient only if the ratio $d_m^{\eta^*}/m$ (stability versus work) increases. We see that this is indeed the case. Comparing with the two 'classical' methods discussed in the beginning of the paper, for which $d_{\mathrm{PL}}/m = 1/2$, $d_{\mathrm{RS}}/m = 0.28$ (with $m = 2$), we see that the S-ROCK has a work/stability ration up to 150 times larger. The last picture of Fig. 3 (lower right plot) gives the optimal values of η for $m \leqslant 200$. Numerical experiments reported in [3] show that (explicit) S-ROCK methods can perform significantly better than classical explicit methods for stiff stochastic problems.

Acknowledgements

Assyr Abdulle is supported by an EPSRC Advanced Fellowship EP/E05207X/1.

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