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Differential Geometry

Compact blow-up limits of finite time singularities of Ricci flow are shrinking Ricci solitons

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Abstract

Using the λ and μ functional introduced by Perelman, we prove that the compact blow-up limit of a Ricci flow which generates singularities at finite time must be a shrinking Ricci soliton. ***To cite this article: Z.-l. Zhang, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Les limites d'explosion compactes en temps fini de singularités du flot de Ricci sont des solitons « rapetissés ». Utilisant les fonctionnelles λ et μ introduites par Perelman, nous démontrons que les limites d'explosion compactes, en temps fini, du flot de Ricci engendrent des singularities de type solitons « rapetissés ». ***Pour citer cet article : Z.-l. Zhang, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

In this Note, we consider the solutions to the following Ricci flow equation on a closed manifold M:

$$\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}\big(g(t)\big), \tag{1}$$

where $\mathrm{Ric}(g(t))$ is the Ricci tensor of the metric $g(t)$. In [1], Hamilton introduced the Ricci flow equation and proved that for any given initial metric $g(0)$, the solution will exists uniquely for all time, unless the curvature blow ups at some finite time.

A *Ricci soliton* is a solution to Eq. (1) such that $g(t)$ changes by diffeomorphisms and rescalings, that's, $g(t) = \alpha(t)\phi(t)^*g_0$ for positive constants $\alpha(t)$ and a family of diffeomorphisms $\phi(t)$. A soliton is called *shrinking, steady* or *expanding*, if $\alpha' < 0, = 0$ or > 0 correspondingly. It's known that a compact steady (expanding resp.) Ricci soliton is Ricci flat (negative Einstein resp.), see [3,5] for example.

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An important distribution of [5] is showing that the Ricci flow is a gradient flow, with its fixed points (modulo diffeomorphisms and rescalings) exactly the Ricci solitons. So, generally, we hope the following theorem holds, and, in fact, we confirm it in this Note:

Theorem 1.1 *(Main theorem). Let $g(t), t \in [0, T)$, be a maximal solution to the Ricci flow equation on a closed manifold M with singular time $T < \infty$. Let $t_k \to T$ be a sequence of times such that $Q_k = |Rm|(p_k, t_k) \to \infty$. If the rescaled sequence $(M, Q_k g(Q_k^{-1} t + t_k))$ converges in the C^∞ sense to a closed ancient solution $(M, g_\infty(t))$ to the Ricci flow, then $g_\infty(t)$ must be a shrinking Ricci soliton.*

Here, $|Rm|(p_k, t_k)$ denotes the norm of Riemannian curvature tensor of $g(t_k)$ at p_k. The proof uses the λ and μ functional in [5]. See Section 1 for a definition and some properties of them. As for the definition and proof of the convergence of Ricci flow solutions, see [2,5].

On the other hand, if $g(t)$ exists for all time and $t_k \to \infty$ is a sequence of times such that $g(t_k)$ converge, modulo the rescalings and diffeomorphisms, to a metric g_∞ on M, then by the monotonicity of λ functional along the Ricci flow [5], g_∞ must be Einstein of negative or zero type. So we have

Theorem 1.2. *Let $g(t), t \in [0, T)$, be a maximal solution to the Ricci flow equation on a closed manifold M. Suppose there is a sequence of times $t_k \to T$ and positive numbers α_k such that the rescaled metrics $\alpha_k g(t_k)$ converge in the C^∞ sense to a metric g_∞ on M, then g_∞ must be a Ricci soliton metric. If $T = \infty$, then g_∞ is in fact Einstein.*

By the classification of three dimensional compact solitons [3], if M is of dimension three, then M itself is diffeomorphic to a space of constant curvature.

We provide some preliminaries about the λ and μ functional in Section 2. Then we prove the main theorem in Section 3.

2. Basics about λ and μ functionals

In this section, we recall some basics about the λ, μ and ν functionals introduced by Perelman [5], and derive some estimate about them which we will use in the next section.

Given a closed Riemannian manifold (M, g), define $\lambda(g) = \inf\{\int_M (R + |\nabla f|^2) \mathrm{e}^{-f} \,\mathrm{d}v \mid \int_M \mathrm{e}^{-f} \,\mathrm{d}v = 1\}$ and for $\tau > 0$, define $\mu(g, \tau) = \inf\{\int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-n/2} \mathrm{e}^{-f} \,\mathrm{d}v \mid \int_M (4\pi\tau)^{-n/2} \mathrm{e}^{-f} \,\mathrm{d}v = 1\}$, where R denotes the scalar curvature of g, the infirmum is taken over functions $f \in C^\infty(M)$. It's known that λ is just the lowest eigenvalue of the operator $-4\Delta + R$. By a result of Rothaus [6], for each $\tau > 0$, there is a smooth minimizer of $\mu(g, \tau)$. The functional ν is defined by $\nu(g) = \inf_{\tau > 0} \mu(g, \tau)$. $\nu(g)$ is always less than 0 and may equal $-\infty$ by the following Corollary 2.2.

Lemma 2.1. *We have the upper bound*

$$\mu(g, \tau) \leqslant \tau\lambda(g) + \mathrm{Vol}(g) - \frac{n}{2} \ln(4\pi\tau) - n, \tag{2}$$

and the lower bound for $\tau > \frac{n}{8}$,

$$\mu(g, \tau) \geqslant \lambda\tau - \frac{n}{2} \ln(4\pi\tau) - n - \frac{n}{8}(\lambda - \inf R) - n \ln C_s, \tag{3}$$

where C_s denotes the Sobolev constant for g such that $\|\phi\|_{L^{\frac{2n}{n-2}}(g)} \leqslant C_s \|\phi\|_{H^{1,2}(g)}$ for all $\phi \in C^\infty(M)$.

Proof. By definition, set $u = (4\pi\tau)^{-n/4} \mathrm{e}^{-f/2}$, then $\int_M u^2 \,\mathrm{d}v = 1$ and so

$$\begin{aligned}\int_M \big[\tau\big(R + |\nabla f|^2\big) + f - n\big](4\pi\tau)^{-n/2} \mathrm{e}^{-f} \,\mathrm{d}v &= \tau \int_M \big(Ru^2 + 4|\nabla u|^2\big) \,\mathrm{d}v - \int_M u^2 \ln u^2 \,\mathrm{d}v - \frac{n}{2} \ln(4\pi\tau) - n \\ &\leqslant \tau \int_M \big(Ru^2 + 4|\nabla u|^2\big) \,\mathrm{d}v + \mathrm{Vol}(g) - \frac{n}{2} \ln(4\pi\tau) - n,\end{aligned}$$

where we used that $-t\ln t \leqslant 1$ for all $t > 0$. The upper bound follows by choosing f such that u is the eigenfunction of the first eigenvalue of $-4\Delta + R$.

As for the lower bound, let $\bar{f}$ be the minimizer of $\mu(g,\tau)$ for fixed $\tau > 0$ and set $\bar{u} = (4\pi\tau)^{-n/4}\mathrm{e}^{-\bar{f}/2}$. We estimate the term $-\int_M \bar{u}^2 \ln \bar{u}^2\,\mathrm{d}v$ by

$$-\int_M \bar{u}^2 \ln \bar{u}^2\,\mathrm{d}v = -\frac{n-2}{2}\int_M \bar{u}^2 \ln \bar{u}^{\frac{4}{n-2}}\,\mathrm{d}v \geqslant -n\ln\|\bar{u}\|_{L^{\frac{2n}{n-2}}(g)} \geqslant -\frac{n}{2}\ln\left(1+\int_M |\nabla\bar{u}|^2\,\mathrm{d}v\right) - n\ln C_s,$$

where we used the Jensen and Sobolev inequality in the first and the second inequality. Then we have

$$\begin{aligned}\mu(g,\tau) &= \tau\int_M \left(R\bar{u}^2 + 4|\nabla\bar{u}|^2\right)\mathrm{d}v - \int_M \bar{u}^2\ln\bar{u}^2\,\mathrm{d}v - \frac{n}{2}\ln(4\pi\tau) - n\\ &\geqslant \tau\int_M \left(R\bar{u}^2 + 4|\nabla\bar{u}|^2\right)\mathrm{d}v - \frac{n}{2}\ln\left(1+\int_M |\nabla\bar{u}|^2\,\mathrm{d}v\right) - \frac{n}{2}\ln(4\pi\tau) - n - n\ln C_s\\ &\geqslant \left(\tau - \frac{n}{8}\right)\int_M \left(R\bar{u}^2 + 4|\nabla\bar{u}|^2\right)\mathrm{d}v + \frac{n}{8}\int_M R\bar{u}^2\,\mathrm{d}v - \frac{n}{2}\ln(4\pi\tau) - n - n\ln C_s,\end{aligned}$$

which proves the lower bound if we set $\tau > \frac{n}{8}$. □

Corollary 2.2. *We have $\nu(g) \neq -\infty$ iff $\lambda(g) > 0$.*

Proof. It is easily seen that $\mu(g,\tau)$ is continuous for $\tau > 0$, since $\mu(g,\tau)$ is always attainable by some smooth function by [6]. Now the result follows from above lemma and Claim 3.1 of [5], which says that $\mu(g,\tau) \to 0$ as $\tau \to 0^+$. □

It follows that ν functional is valuable only when $\lambda > 0$. Fortunately, the assumption of our main theorem implies the positivity of λ along the Ricci flow. This fact will be proved in the next section.

Corollary 2.3. *If $\lambda(g) \leqslant 0$, then $\mu(g,\tau) \leqslant \ln \mathrm{Vol}(g) - \frac{n}{2}\ln(4\pi\tau) - n + 1$.*

Proof. First note that $\mu(\alpha g, \alpha\tau) = \mu(g,\tau)$ for any $\alpha > 0$ by a direct computation. Set $Q = \mathrm{Vol}(g)^{-2/n}$, then by the above lemma

$$\mu(g,\tau) = \mu(Qg, Q\tau) \leqslant Q\tau\lambda(Qg) - \frac{n}{2}\ln(4\pi Q\tau) - n + \mathrm{Vol}(Qg) \leqslant \ln\mathrm{Vol}(g) - \frac{n}{2}\ln(4\pi\tau) - n + 1. \qquad \square$$

Finally, we state a property about the monotonicity of ν functional along the Ricci flow, due to Perelman [5,4]:

Proposition 2.4. *Let $g(t)$ be a solution to the Ricci flow equation* (1) *on a closed manifold M. Denote $\tau(t) = T - t$ for some constant $T > 0$, then $\mu(g(t),\tau(t))$ is non-decreasing whenever it makes sense. Moreover, the monotonicity is strict unless $g(t)$ is a shrinking Ricci soliton.*

If $\lambda(g(0)) > 0$, then $\nu(g(t))$ increases strictly unless $g(t)$ is a shrinking Ricci soliton.

3. Compact shrinking Ricci solitons as blow-up limits

In this section we will prove our main theorem. The idea is to use the monotonicity of ν functional along the Ricci flow. We begin with a lemma:

Lemma 3.1. *Let $g(t)$, $t \in [0,T)$, be a solution to the Ricci flow equation* (1) *on a closed manifold M. If $\lambda(g(t)) \leqslant 0$ for all t, then there exist constants $c_1, c_2 > 0$ depending only on $g(0)$, such that for all $t \geqslant 0$ we have* $\mathrm{Vol}(g(t)) \geqslant c_1\mathrm{e}^{-c_2 t}$.

Proof. By Proposition 2.4 and Lemma 2.1, we have

$$\mu\left(g(t),\frac{n}{8}\right) \geqslant \mu\left(g(0),\frac{n}{8}+t\right) \geqslant \lambda\big(g(0)\big)t - \frac{n}{2}\ln\left(4\pi\left(t+\frac{8}{n}\right)\right) + \frac{n}{8}\inf R(\cdot,0) - n - n\ln C_s\big(g(0)\big)$$
$$\geqslant \left(\lambda\big(g(0)\big) - \frac{n^2}{16}\right)t - \frac{n}{2}\ln\left(\frac{32}{n}\pi\right) + \frac{n}{8}\inf R(\cdot,0) - n - n\ln C_s\big(g(0)\big),$$

where $C_s(g(0))$ denotes the Sobolev constant of $(M, g(0))$. Setting $c_1 = \exp(\frac{n}{8}\inf R(\cdot,0) + \frac{n}{2}\ln\frac{n^2}{64} - n\ln C_s(g(0)) - 1)$ and $c_2 = -\lambda(g(0)) + \frac{n^2}{16}$, and substituting $\tau = \frac{n}{8}$ into Corollary 2.3, we obtain the estimate $\mathrm{Vol}(g(t)) \geqslant \exp(\mu(g(t),\frac{n}{8}) + \frac{n}{2}\ln(\frac{n}{2}\pi) + n - 1) \geqslant c_1\exp(-c_2 t)$. □

Corollary 3.2. *Let $g(t)$, $t \in [0,T)$, be a maximal solution to the Ricci flow equation* (1) *on a closed manifold M with $T < \infty$. If $\lambda(g(t)) \leqslant 0$ for all t, then any blow-up limit is noncompact.*

Proof. Suppose we have a blow-up sequence $(M, Q_k g(Q_k^{-1}t + t_k), p_k)$ of Ricci flow solutions with $Q_k \to \infty$. By assumption and above lemma, we have that the rescaled volume at time zero equals $Q_k^{n/2}\mathrm{Vol}(g(t_k)) \to \infty$. So the limit has infinite volume and consequently can't be compact. □

Now we are ready to give a

Proof of the Main theorem. By above corollary, we may assume that $\lambda(g(0)) > 0$. So Proposition 2.4 uses and there is a limit $\sigma = \lim_{t\to T^-}\nu(g(t)) \leqslant 0$. Then for any $t \in (-\infty, 0]$, by the smooth convergence,

$$\nu\big(g_\infty(t)\big) = \lim_{k\to\infty}\nu\big(Q_k g\big(Q_k^{-1}t + t_k\big)\big) = \lim_{k\to\infty}\nu\big(g\big(Q_k^{-1}t + t_k\big)\big) = \lim_{t\to T^-}\nu\big(g(t)\big) = \sigma.$$

That is, the ν functional is constant on the limit flow. Then Corollary 2.2 and Proposition 2.4 imply that $g_\infty(t)$ must be a shrinking Ricci soliton. □

Proof of Theorem 1.2. By our main theorem, it suffice to consider the case $T = \infty$. By Proposition 1.2 of [5], $\lambda(g(t)) \leqslant 0$ for all time. Denote by $\bar\lambda(g(t)) = \lambda(g(t))\mathrm{Vol}(g(t))^{2/n}$ the normalized λ value along the Ricci flow, then by Claim 2.3 of [5], $\bar\lambda(g(t))$ increases in t and stays constant only when $g(t)$ is Einstein of negative or zero type. The curvature of $\alpha_k g(t_k)$ is uniformly bounded since $\alpha_k g(t_k) \to g_\infty$ smoothly. So from the evolution of the curvature $\frac{\partial}{\partial t}|Rm|^2 \leqslant \Delta|Rm|^2 + C|Rm|^2$, the Ricci flow $g_k(t) = \alpha_k g(\alpha_k^{-1}t + t_k)$ exists on a time interval $[0,\delta]$ for δ independent of k. Now Hamilton's compactness theorem for Ricci flow solutions [2] shows that $g_k(t) \to g_\infty(t)$, $t \in [0,\delta)$, in C^∞ sense. But $\bar\lambda(g_\infty(t)) \equiv \lim_{t\to\infty}\bar\lambda(g(t))$ is a constant $\leqslant 0$, so g_∞ must be Einstein of negative or zero type by arguments in [5]. □

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