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Probability Theory

A remarkable σ-finite measure on $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$ related to many Brownian penalisations

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Abstract

In this Note, we study a σ-finite measure $\mathcal{W}$ on the space $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$, strongly related to Wiener measure, and we construct a large class of Brownian martingales from $\mathcal{W}$. Some of these martingales appear naturally in the study of Brownian penalisations made by B. Roynette, P. Vallois and M. Yor. ***To cite this article: J. Najnudel et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Une mesure σ-finie sur $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$ liée à de nombreuses pénalisations browniennes. Dans cette Note, nous étudions une mesure σ-finie $\mathcal{W}$ sur l'espace $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$, fortement liée à la mesure de Wiener, et nous construisons une grande classe de martingales browniennes à partir de $\mathcal{W}$. Certaines de ces martingales apparaissent naturellement dans l'étude des pénalisations browniennes faite par B. Roynette, P. Vallois et M. Yor. ***Pour citer cet article : J. Najnudel et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Dans de nombreux cas de pénalisation de la mesure de Wiener $\mathbf{W}$ étudiés par B. Roynette, P. Vallois et M. Yor (voir [5–7]), la mesure limite obtenue est absolument continue par rapport à la mesure σ-finie $\mathcal{W}$ définie sur l'espace $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$ par la formule :

$$\mathcal{W} = \int_0^\infty \mathrm{d}l \left(\mathbf{W}^{\tau_l} \circ \mathbf{P}^{(3),\mathrm{sym}}\right), \tag{1}$$

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où $\mathbf{W}^{\tau_l}$ est la loi d'un mouvement brownien standard arrêté en τ_l, $(\tau_u)_{u\geqslant 0}$ étant l'inverse de son temps local au niveau zéro, où

$$\mathbf{P}^{(3),\mathrm{sym}} = \frac{1}{2}\big(\mathbf{P}^{(3)} + \widetilde{\mathbf{P}}^{(3)}\big), \tag{2}$$

$\mathbf{P}^{(3)}$ (resp. $\widetilde{\mathbf{P}}^{(3)}$) désignant la loi d'un processus de Bessel (resp. de l'opposé d'un processus de Bessel) de dimension 3 issu de 0, et où $(\mathbf{W}^{\tau_l} \circ \mathbf{P}^{(3),\mathrm{sym}})$ est la concaténation des probabilités $\mathbf{W}^{\tau_l}$ et $\mathbf{P}^{(3),\mathrm{sym}}$ (voir [1] pour cette dernière notation).

La mesure $\mathcal{W}$ vérifie également l'équation :

$$\mathcal{W}[F_t \mathbf{1}_{\gamma_a \leqslant t}] = \mathbf{W}\big[F_t |X_t - a|\big] \tag{3}$$

où X est le processus canonique de $\mathcal{C}(\mathbf{R}_+,\mathbf{R})$, $\gamma_a = \sup\{t \geqslant 0, X_t = a\}$ et $F_t \geqslant 0$ est mesurable par rapport à la tribu $\mathcal{F}_t = \sigma\{X_s, s \leqslant t\}$.

Cette formule permet de montrer que pour toute fonctionnelle positive Φ telle que $\mathcal{W}(\Phi) < \infty$, le processus $\mathcal{F}_t$-adapté $(M_t(\Phi))_{t\geqslant 0}$ défini par :

$$M_t(\Phi)\big((X_s)_{s\leqslant t}\big) = \int_{\mathcal{C}(\mathbf{R}_+,\mathbf{R})} \Phi\big((X_s)_{s\leqslant t} \circ (X_t + Y_u)_{u\geqslant 0}\big)\mathcal{W}(\mathrm{d}Y) \tag{4}$$

est une martingale sous $\mathbf{W}$; de plus, cette martingale est la densité de la restriction à $\mathcal{F}_t$ de la mesure $\Phi.\mathcal{W}$, par rapport à $\mathbf{W}_{|\mathcal{F}_t}$:

$$(\Phi.\mathcal{W})_{|\mathcal{F}_t} = M_t(\Phi).\mathbf{W}_{|\mathcal{F}_t}. \tag{5}$$

On remarque aussi que si F_t est une fonctionnelle positive et $\mathcal{F}_t$-mesurable, $\mathcal{W}(F_t) = 0$ ou $\mathcal{W}(F_t) = \infty$: ainsi, $\mathcal{W}$ n'est σ-finie sur aucune des tribus $\mathcal{F}_t$, $t > 0$.

La mesure $\mathcal{W}$ intervient également dans l'étude des mesures invariantes pour le processus de Markov $(\mathcal{X}_t)_{t\in\mathbf{R}_+}$ à valeurs dans $\mathcal{C}(\mathbf{R}_+,\mathbf{R})$, donné par

$$\mathcal{X}_t(u) = \begin{cases} \mathcal{X}_0(u-t), & \text{si } u \geqslant t, \\ \mathcal{X}_0(0) + B_{t-u}, & \text{si } u \leqslant t, \end{cases} \tag{6}$$

$\mathcal{X}_0$ étant une variable aléatoire fixée à valeurs dans $\mathcal{C}(\mathbf{R}_+,\mathbf{R})$, et B étant un mouvement brownien standard, indépendant de $\mathcal{X}_0$.

Les preuves des résultats annoncés dans cette Note sont détaillées dans [3].

1. Existence of the measure $\mathcal{W}$

In this Note, $\mathbf{W}$ denotes Wiener measure on $\mathcal{C}(\mathbf{R}_+,\mathbf{R})$, X is the canonical process on this space, and for all $s \in \mathbf{R}_+$, $\mathcal{F}_s$ is the σ-field generated by X_u, $u \leqslant s$.

Moreover, for $x \in \mathbf{R}$, $t \in \mathbf{R}_+$, L_t^x denotes the local time of X at time t and level x.

In recent articles, B. Roynette, P. Vallois and M. Yor (see [8] for a synthesis of the results of these articles) have proven that for several families of functionals $\Gamma = (\Gamma_t)_{t\in\mathbf{R}_+}$ from $\mathcal{C}(\mathbf{R}_+,\mathbf{R})$ to $\mathbf{R}_+$, there exists a limit probability measure $\mathbf{W}_\infty^\Gamma$ on $\mathcal{C}(\mathbf{R}_+,\mathbf{R})$ such that, for every bounded and $\mathcal{F}_s$-measurable functional F_s ($s \geqslant 0$):

$$\mathbf{W}_\infty^\Gamma(F_s) = \lim_{t\to\infty} \frac{\mathbf{W}[F_s \Gamma_t]}{\mathbf{W}[\Gamma_t]}. \tag{7}$$

We say that $(\Gamma_t)_{t\in\mathbf{R}_+}$ is a penalisation process, and $\mathbf{W}_\infty^\Gamma$ is $\mathbf{W}$, penalised by Γ.

For example, the above convergence holds if:

$$\Gamma_t = \Gamma_t^{(q)} = \exp\left(-\frac{1}{2}A_t^{(q)}\right) \tag{8}$$

where

$$A_t^{(q)} = \int_{-\infty}^{\infty} q(\mathrm{d}x) L_t^x, \tag{9}$$

and q is a positive measure on $\mathbf{R}$ such that

$$0 < \int_{-\infty}^{\infty} q(\mathrm{d}x)\big(1+|x|\big) < \infty. \tag{10}$$

More precisely, if $\mathbf{W}_x$ denotes Wiener measure starting at x, there exists a function $\varphi_q : \mathbf{R} \to \mathbf{R}_+$ such that:

$$\varphi_q(x) = \lim_{t\to\infty} \sqrt{\frac{\pi t}{2}} \mathbf{W}_x\left[\exp\left(-\frac{1}{2} A_t^{(q)}\right)\right] \tag{11}$$

and the limit measure $\mathbf{W}_\infty^{(q)} := \mathbf{W}_\infty^{\Gamma^{(q)}}$ satisfies the absolute continuity relation:

$$\mathbf{W}_{\infty\ |\mathcal{F}_t}^{(q)} = \frac{\varphi_q(X_t)}{\varphi_q(0)} \exp\left(-\frac{1}{2} A_t^{(q)}\right).\mathbf{W}_{|\mathcal{F}_t}. \tag{12}$$

Now, it may be proven that the following measure:

$$\mathcal{W} = \varphi_q(0) \exp\left(\frac{1}{2} A_t^{(q)}\right).\mathbf{W}_\infty^{(q)} \tag{13}$$

does not depend on q (see, for example, [9]).

Moreover, since

$$\mathbf{W}_\infty^{(q)} = \frac{\exp(-\frac{1}{2} A_\infty^{(q)})}{\varphi_q(0)}.\mathcal{W}, \tag{14}$$

one has:

$$\mathbf{W}_\infty^{(q)} = \frac{\Gamma_\infty^{(q)}}{\mathcal{W}[\Gamma_\infty^{(q)}]}.\mathcal{W} \tag{15}$$

where $\Gamma_\infty^{(q)} = \lim_{t\to\infty} \Gamma_t^{(q)}$.

In fact, there are many other penalisation processes Γ such that:

$$\mathbf{W}_\infty^{\Gamma} = \frac{\Gamma_\infty}{\mathcal{W}[\Gamma_\infty]}.\mathcal{W} \tag{16}$$

where $\Gamma_\infty = \lim_{t\to\infty} \Gamma_t$; for example, if h is a positive and integrable function, $\Gamma_t^{(1)} = h(\sup_{[0,t]} X)$ and $\Gamma_t^{(2)} = h(L_t^0)$ are two such penalisation processes.

For more general examples, see [2].

As a consequence of formula (16), penalisation studies are closely related with properties of $\mathcal{W}$ established below in this Note.

2. Other descriptions of $\mathcal{W}$

Let us denote by $\mathbf{W}^{\tau_l}$ the law of a standard Brownian motion stopped at its inverse local time τ_l at level 0.

Denoting by $\mathbf{P}^{(3),\mathrm{sym}}$ the measure defined by the formula:

$$\mathbf{P}^{(3),\mathrm{sym}} = \frac{1}{2}\big(\mathbf{P}^{(3)} + \widetilde{\mathbf{P}}^{(3)}\big) \tag{17}$$

where $\mathbf{P}^{(3)}$ is the law of a standard Bessel process of dimension 3, and $\widetilde{\mathbf{P}}^{(3)}$ is the law of the opposite of a BES(3) process, the measure $\mathcal{W}$ may be described by the following formula (denoted by (1) at the beginning of this Note):

$$\mathcal{W} = \int_0^\infty \mathrm{d}l\big(\mathbf{W}^{\tau_l} \circ \mathbf{P}^{(3),\mathrm{sym}}\big) \tag{18}$$

where $\circ$ denotes, at the level of measures, the operation of concatenation (see, for example, [1] for these notations).

Furthermore, in [1] and [4] (Chap. XII.4), the following identity is proven:

$$\int_0^\infty \mathrm{d}l\, \mathbf{W}^{\tau_l} = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{2\pi t}} \mathbf{\Pi}_{0,0}^t \tag{19}$$

where $\mathbf{\Pi}_{0,0}^t$ is the law of the standard Brownian bridge on the interval $[0,t]$.

Hence:

$$\mathcal{W} = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{2\pi t}} \big(\mathbf{\Pi}_{0,0}^t \circ \mathbf{P}^{(3),\mathrm{sym}}\big). \tag{20}$$

These expressions of $\mathcal{W}$ imply the following proposition:

Proposition 2.1. *If g denotes the last hitting time at zero of X, one has, for $l \geqslant 0$, $u > 0$:*

$$\mathcal{W}\big(L_\infty^0 \in \mathrm{d}l,\, g \in \mathrm{d}u\big) = \frac{l e^{-l^2/2u}}{\sqrt{2\pi u^3}}\, \mathrm{d}l\, \mathrm{d}u, \tag{21}$$

$$\mathcal{W}\big(L_\infty^0 - L_t^0 \in \mathrm{d}l\big) = \mathrm{d}l + \sqrt{\frac{2t}{\pi}}\, \delta_0(\mathrm{d}l), \tag{22}$$

where for $a \in \mathbf{R}$, δ_a is Dirac measure at a.

Moreover, conditionally on $L_\infty^0 - L_t^0 = l$ $(l > 0)$, $(X_u)_{u \leqslant t}$ under $\mathcal{W}$ is a Brownian motion on $[0,t]$.

Note that formula (21) given in Proposition 2.1 implies in particular:

$$\mathcal{W}(g \in \mathrm{d}t) = \mathrm{d}t/\sqrt{2\pi t}, \tag{23}$$

$$\mathcal{W}\big(L_\infty^0 \in \mathrm{d}l\big) = \mathrm{d}l. \tag{24}$$

Another interesting formula is the following: for all positive and $\mathcal{F}_t$-measurable functional F_t, and for all $a \in \mathbf{R}$:

$$\mathcal{W}[F_t \mathbf{1}_{\gamma_a \leqslant t}] = \mathbf{W}\big[F_t |X_t - a|\big] \tag{25}$$

where $\gamma_a = \sup\{t \geqslant 0,\, X_t = a\}$.

One has also:

$$\mathcal{W}\bigg[F_t \exp\bigg(-\frac{1}{2}\big(A_\infty^{(q)} - A_t^{(q)}\big)\bigg)\bigg] = \mathbf{W}\big[F_t \varphi_q(X_t)\big] \tag{26}$$

if q satisfies the above conditions (10). Note that formula (25) can be formally considered to be (26) for $q = \infty.\delta_a$.

Remark. The measure $\mathcal{W}$ is σ-finite on $\mathcal{F}_\infty = \sigma\{X_s, s \in \mathbf{R}_+\}$, but it is not σ-finite on any $\mathcal{F}_t$ $(t > 0)$: for every $\mathcal{F}_t$-measurable and positive functional F_t, $\mathcal{W}(F_t) = 0$ or $\mathcal{W}(F_t) = \infty$.

3. On Brownian martingales generated by the measure $\mathcal{W}$

We define here an important class of $(\mathbf{W}, (\mathcal{F}_t)_{t \geqslant 0})$-martingales, which can be constructed from the measure $\mathcal{W}$.

More precisely, let Φ be in $L_+^1(\mathcal{W})$; the measure $\mathbf{W}^\Phi = \Phi.\mathcal{W}$ is absolutely continuous with respect to $\mathbf{W}$ on each σ-field $\mathcal{F}_t$, so we can define a martingale $(M_t(\Phi))_{t \geqslant 0}$ by the equality:

$$\mathbf{W}^\Phi_{|\mathcal{F}_t} = M_t(\Phi).\mathbf{W}_{|\mathcal{F}_t}. \tag{27}$$

Moreover, it is possible to 'compute' $M_t(\Phi)$ by the formula:

$$M_t(\Phi)\big((X_s)_{s \leqslant t}\big) = \int_{\mathcal{C}(\mathbf{R}_+,\mathbf{R})} \Phi\big((X_s)_{s \leqslant t} \circ (X_t + Y_u)_{u \geqslant 0}\big) \mathcal{W}(\mathrm{d}Y) \tag{28}$$

where $\circ$ denotes the operation of concatenation of trajectories.

In particular:

$$M_0(\Phi) = \int\limits_{\mathcal{C}(\mathbf{R}_+,\mathbf{R})} \Phi(Y)\mathcal{W}(\mathrm{d}Y) = \mathcal{W}(\Phi) \tag{29}$$

is the total mass of $\mathbf{W}^{\Phi}$.

The family $\mathcal{M}$ of martingales $(M(\Phi))_{\Phi \in L^1_+(\mathcal{W})}$ satisfies the following identities:

(i) For $\Phi, \Psi \in L^1_+(\mathcal{W})$, F_t bounded and $\mathcal{F}_t$-measurable:

$$\mathbf{W}^{\Psi}\big(F_t M_t(\Phi)\big) = \mathbf{W}^{\Phi}\big(F_t M_t(\Psi)\big) = \mathbf{W}\big[F_t M_t(\Phi) M_t(\Psi)\big]. \tag{30}$$

(ii) For $\Phi, \Psi \in L^1_+(\mathcal{W})$ and $\Psi > 0$:

$$\mathbf{W}^{\Psi}\left(\frac{\Phi}{\Psi}\middle|\mathcal{F}_t\right) = \frac{M_t(\Phi)}{M_t(\Psi)}. \tag{31}$$

Moreover, there exists a kind of inverse of the above construction: it is possible to retrieve $\Phi \in L^1_+(\mathcal{W})$ from $M(\Phi)$ by the formula

$$\frac{M_t(\Phi)}{1+|X_t|} \underset{t\to\infty}{\overset{\mathcal{W}-\text{a.s.}}{\longrightarrow}} \Phi, \tag{32}$$

which means that if Δ denotes the set of trajectories $X \in \mathcal{C}(\mathbf{R}_+, \mathbf{R})$ such that $\frac{M_t(\Phi)(X)}{1+|X_t|}$ does not tend to $\Phi(X)$, then $\mathcal{W}(\Delta) = 0$.

More generally, the following proposition holds:

Proposition 3.1. *Let $(M_t)_{t\geqslant 0}$ denote a positive $(\mathbf{W}, (\mathcal{F}_t)_{t\in\mathbf{R}_+})$-supermartingale.*

(1) *$M_\infty = \lim_{t\to\infty} M_t$ exists $\mathbf{W}$-a.s. and $\mathbf{W}(M_\infty) < \infty$.*
(2) *$\mu_\infty = \lim_{t\to\infty} \frac{M_t}{1+|X_t|}$ exists $\mathcal{W}$-a.s. and $\mathcal{W}(\mu_\infty) < \infty$.*
(3) *There is the decomposition (for $t \in \mathbf{R}_+$):*

$$M_t = M_t(\mu_\infty) + \mathbf{W}(M_\infty|\mathcal{F}_t) + \xi_t, \tag{33}$$

where $(\xi_t)_{t\in\mathbf{R}_+}$ is a $(\mathbf{W}, (\mathcal{F}_t)_{t\in\mathbf{R}_+})$-supermartingale such that:

$$M_t(\mu_\infty) + \xi_t \underset{t\to\infty}{\overset{\mathbf{W}\text{-a.s.}}{\longrightarrow}} 0, \tag{34}$$

$$\frac{\mathbf{W}(M_\infty|\mathcal{F}_t) + \xi_t}{1+|X_t|} \underset{t\to\infty}{\overset{\mathcal{W}\text{-a.s.}}{\longrightarrow}} 0. \tag{35}$$

(4) *The formula (33) is the unique decomposition of the form:*

$$M_t = M_t(\Phi) + \mathbf{W}(Z|\mathcal{F}_t) + \zeta_t, \tag{36}$$

where $\Phi \in L^1_+(\mathcal{W})$, $Z \in L^1_+(\mathbf{W})$ and $(\zeta_t)_{t\in\mathbf{R}_+}$ is a supermartingale such that:

$$\zeta_t \underset{t\to\infty}{\overset{\mathbf{W}\text{-a.s.}}{\longrightarrow}} 0, \tag{37}$$

$$\frac{\zeta_t}{1+|X_t|} \underset{t\to\infty}{\overset{\mathcal{W}\text{-a.s.}}{\longrightarrow}} 0. \tag{38}$$

Of course, if $(M_t)_{t\in\mathbf{R}_+}$ is a martingale, $(\xi_t)_{t\in\mathbf{R}_+}$ in (33) is also a martingale.

Proposition 3.1 implies the following:

Corollary 3.2. *A positive* $(\mathbf{W}, (\mathcal{F}_t)_{t\in\mathbf{R}_+})$*-martingale is equal to* $M(\Phi)$ *for a functional* $\Phi \in L^1_+(\mathcal{W})$ *if and only if*:

$$M_0 = \mathcal{W}\left(\lim_{t\to\infty} \frac{M_t}{1+|X_t|}\right). \tag{39}$$

Note that $\lim_{t\to\infty} \frac{M_t}{1+|X_t|}$ *exists* $\mathcal{W}$*-a.s. by Proposition* 3.1.

Examples of martingales in $\mathcal{M}$**.**

(a) If $\Phi = h(L^0_\infty)$ where $h : \mathbf{R}_+ \to \mathbf{R}_+$ is integrable, Eq. (24) implies that $\mathcal{W}(\Phi) < \infty$, and one has:

$$M_t(\Phi) = h\big(L^0_t\big)|X_t| + \int_{L^0_t}^{\infty} h(x)\,\mathrm{d}x. \tag{40}$$

(b) If $\Phi = \exp(-\frac{1}{2}A^{(q)}_\infty)$, where $0 < \int_{-\infty}^{\infty} q(\mathrm{d}x)(1+|x|) < \infty$:

$$M_t(\Phi) = \varphi_q(X_t)\exp\left(-\frac{1}{2}A^{(q)}_t\right). \tag{41}$$

We remark that for $q = \lambda.\delta_0$ and $h(l) = \mathrm{e}^{-\frac{\lambda}{2}l}$, the martingales (a) and (b) coincide and are equal to $\mathrm{e}^{-\frac{\lambda L^0_t}{2}}(\frac{2}{\lambda} + |X_t|)$.

(c) If $\Phi = h(g)$, where $g = \sup\{t, X_t = 0\}$, $h \geqslant 0$ and $0 < \int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}} h(t) < \infty$:

$$M_t(\Phi) = h(g_t)|X_t| + \int_0^\infty \frac{\mathrm{e}^{-X_t^2/2u}}{\sqrt{2\pi u}} h(t+u)\,\mathrm{d}u \tag{42}$$

for $g_t = \sup\{s \in [0,t], X_s = 0\}$.

4. Ray–Knight theorem for $\mathcal{W}$

For $x \in \mathbf{R}$, let $\mathcal{W}_x$ be the image of $\mathcal{W}$ by translation of x; i.e. for every positive and measurable functional F:

$$\mathcal{W}_x\big[F\big((X_s)_{s\in\mathbf{R}_+}\big)\big] = \mathcal{W}\big[F\big((X_s + x)_{s\in\mathbf{R}_+}\big)\big]. \tag{43}$$

It is possible, by using Ray–Knight theorems both for Brownian motion stopped at an inverse local time and for BES(3) process, to describe the 'law' of the total local times of the canonical process under the σ-finite measure $\mathcal{W}_x$.

More precisely, for $l, a, b \in \mathbf{R}_+$ and $x \in \mathbf{R}$, let $\mathbf{Q}^{a,b}_{x,l}$ be the law of a process $(Z_u)_{u\in\mathbf{R}}$ such that $Z_x = l$, $(Z_{x+v})_{v\geqslant 0}$ is a squared a-dimensional Bessel process, and $(Z_{x-v})_{v\geqslant 0}$ is an independent BESQ(b) process.

Using these notations, then with the help of formula (18) and additivity property of squared Bessel processes, the image of $\mathcal{W}_x$ by the application 'total local time' $\mathcal{L} : (X_s)_{s\in\mathbf{R}_+} \to (L^y_\infty)_{y\in\mathbf{R}}$ is the σ-finite measure $\mathbf{\Lambda}_x$ on $\mathcal{C}(\mathbf{R}, \mathbf{R}_+)$ described by the following formula:

$$\mathbf{\Lambda}_x = \frac{1}{2}\int_0^\infty \mathrm{d}l\big(\mathbf{Q}^{0,2}_{x,l} + \mathbf{Q}^{2,0}_{x,l}\big). \tag{44}$$

In particular, by using invariance property of Lebesgue measure for BESQ(0) and BESQ(2) process, it is possible to check from formula (44) that the 'law' of total local time of $(X_s)_{s\in\mathbf{R}_+}$ at level y, under $\mathcal{W}_x$, is:

$$\mu_{x,y} = |y-x|\delta_0 + \nu \tag{45}$$

where δ_0 is Dirac measure at zero and ν is Lebesgue measure.

Formula (45) can also be obtained from the martingale properties of $\mathcal{W}$.

5. On invariant measures related to $\mathcal{W}_x$ and $\mathbf{\Lambda}_x$

The measure $\mathcal{W}$ is strongly related to invariant measures of Markov processes on functional spaces.

For example, if $\mathcal{X}_0$ is a given random function in $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$, and B is a standard Brownian motion, let $(\mathcal{X}_t)_{t \in \mathbf{R}_+}$ be the process on $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$, defined by:

$$\mathcal{X}_t(u) = \begin{cases} \mathcal{X}_0(u-t), & \text{if } u \geqslant t, \\ \mathcal{X}_0(0) + B_{t-u}, & \text{if } u \leqslant t. \end{cases} \tag{46}$$

By standard properties of Brownian motion, it is not difficult to check that $\mathcal{X}$ is a Markov process, and that:

$$\widetilde{\mathbf{W}} = \int_{\mathbf{R}} \mathrm{d}x \, \mathbf{W}_x \tag{47}$$

is an invariant measure for $\mathcal{X}$.

Now, a remarkable point is that, because of the martingale properties of $\mathcal{W}$ given in Section 3, the measure:

$$\widetilde{\mathcal{W}} = \int_{\mathbf{R}} \mathrm{d}x \, \mathcal{W}_x \tag{48}$$

is also invariant for $\mathcal{X}$.

We now consider the image $\mathcal{Y}$ of the process $\mathcal{X}$ by the application H from $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$ to $\mathbf{R} \times \mathcal{C}(\mathbf{R}, \mathbf{R}_+)$, defined by:

$$H\big[(X_s)_{s \in \mathbf{R}_+}\big] = \big(X_0, (L_\infty^y)_{y \in \mathbf{R}}\big) \tag{49}$$

if the family $(L_\infty^y)_{y \in \mathbf{R}}$ of total local times of X is well-defined and continuous, and by:

$$H\big[(X_s)_{s \in \mathbf{R}_+}\big] = (0, 0) \tag{50}$$

in the other cases.

It is easy to show that $\mathcal{Y}$ is a Markov process on $\mathbf{R} \times \mathcal{C}(\mathbf{R}, \mathbf{R}_+)$; moreover, since the measure:

$$\tilde{\mathbf{\Lambda}} = \mathrm{d}x \otimes \mathbf{\Lambda}_x \tag{51}$$

is the image of $\widetilde{\mathcal{W}}$ by H, it is an invariant measure for the process $\mathcal{Y}$.

Some open problems are the existence of invariant σ-finite measures for $\mathcal{X}$, which are not convex combinations of $\widetilde{\mathbf{W}}$ and $\widetilde{\mathcal{W}}$, and the existence of invariant σ-finite measures for $\mathcal{Y}$, which are not proportional to $\tilde{\mathbf{\Lambda}}$.

6. A general theorem of penalisation

The measure $\mathcal{W}$ was constructed in order to explain globally some penalisation results which were proven before by B. Roynette, P. Vallois and M. Yor.

In turn, it is possible to generalize these results by using properties of $\mathcal{W}$; more precisely, the following proposition holds:

Proposition 6.1. *Let $(\Gamma_t)_{t \in \mathbf{R}_+}$ be a family of functionals from $\mathcal{C}(\mathbf{R}_+, \mathbf{R})$ to $\mathbf{R}_+$ such that:*

(1) *For all $t \in \mathbf{R}_+$, Γ_t is $\mathcal{F}_t$-measurable.*
(2) *Γ_t decreases to a functional Γ when t goes to infinity.*
(3) *There exists $a \in \mathbf{R}_+$ such that the following property holds: if $\omega \in \mathcal{C}(\mathbf{R}_+, \mathbf{R})$, $\Gamma_t(\omega) = \Gamma(\omega)$ for every $t \geqslant \sigma_a$, where $\sigma_a = \sup\{u \geqslant 0, |\omega(u)| \leqslant a\}$.*
(4) $0 < \mathcal{W}(\Gamma) < \infty$.

In these conditions, for all $s \in \mathbf{R}_+$, and for every bounded, $\mathcal{F}_s$-measurable functional F_s:

$$\frac{\mathbf{W}[F_s \Gamma_t]}{\mathbf{W}[\Gamma_t]} \underset{t \to \infty}{\longrightarrow} \mathbf{W}_\infty^\Gamma(F_s) \tag{52}$$

where the probability $\mathbf{W}_\infty^\Gamma$ *is defined by the equality*:

$$\mathbf{W}_\infty^\Gamma = \frac{\Gamma}{\mathcal{W}(\Gamma)}.\mathcal{W}. \tag{53}$$

Example. If $q:\mathbf{R} \to \mathbf{R}_+$ is a continuous, bounded function with compact support, and if q is not identical to zero,

$$\Gamma_t = \exp\left(-\frac{1}{2}\int_0^t q(X_s)\,\mathrm{d}s\right) \tag{54}$$

satisfies the conditions of Proposition 6.1.

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