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Partial Differential Equations

The Dirac equation on the Anti-de-Sitter Universe

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Abstract

We investigate global solutions of the Dirac equation on the Anti-de-Sitter Universe. Since this space is not globally hyperbolic, the Cauchy problem is not, a priori, well-posed. Nevertheless, this is the case when the mass of the field is large compared to the cosmological constant. In opposite, for the light fermions, we construct several asymptotic conditions at infinity, such that the problem becomes well-posed. In all the cases, the spectrum of the Hamiltonian is discrete. We also get a result of equipartition of the energy. ***To cite this article: A. Bachelot, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

L'équation de Dirac sur l'univers Anti-de Sitter. Nous cherchons des solutions globales de l'équation de Dirac dans l'univers Anti-de Sitter. Comme cet espace n'est pas globalement hyperbolique, le problème de Cauchy n'est pas, a priori, bien posé. Nous montrons que c'est toutefois le cas quand la masse du champ est grande par rapport à la constante cosmologique. En revanche, pour les faibles masses, nous construisons diverses conditions asymptotiques à l'infini, rendant le problème bien posé. Dans tous les cas, l'hamiltonien a un spectre discret. On établit également un résultat d'équipartition de l'énergie. ***Pour citer cet article : A. Bachelot, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Étant donné $\Lambda > 0$, l'univers Anti de Sitter est la variété lorentzienne $AdS = \mathbb{R}_t \times \mathbb{B}$,

$$\mathrm{d}s^2_{AdS} = \left(\frac{1+\varrho^2}{1-\varrho^2}\right)^2 \mathrm{d}t^2 - \frac{3}{\Lambda}\,\mathrm{d}s^2_{\mathbb{B}}$$

où

$$\mathbb{B} := \left\{\mathbf{x} = \varrho\omega = (x^1, x^2, x^3) \in \mathbb{R}^3,\ \varrho \in [0,1[,\ \omega \in S^2\right\}$$

est la boule de Poincaré munie de sa métrique hyperbolique $\mathrm{d}s^2_{\mathbb{B}} = \frac{4}{1-\varrho^2}(\mathrm{d}\varrho^2 + \varrho^2\,\mathrm{d}\omega^2)$. Cet espace n'est pas globalement hyperbolique car les géodésiques nulles issues de l'origine forment un cône dont la frontière est asymptote à

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$\{t=\pm\frac{\pi}{2}\sqrt{\frac{\Lambda}{3}}\}\times\mathbb{B}$. Ceci suggère que le problème de Cauchy global n'est a priori pas bien posé, même pour des champs linéaires, et qu'il faille ajouter une condition à l'infini $\rho=1$. Toutefois, le fait que les géodésiques temporelles soient paramétrisables par $(t,\mathbf{x}(t))_{t\in\mathbb{R}}$, où $\mathbf{x}(t)$ est une fonction t-périodique, invite à penser qu'aucune contrainte à l'infini n'est nécessaire pour les champs massifs. La situation est en fait assez subtile, et dépend de la masse du champ $M>0$, et de la constante cosmologique Λ. Étant donné $\Psi_0\in\mathbf{L}^2:=[L^2(\mathbb{B},\frac{2}{1+\varrho^2}\,\mathrm{d}\mathbf{x})]^4$, nous cherchons $\Psi\in C^0(\mathbb{R}_t;\mathbf{L}^2)$ solution de l'équation de Dirac sur *AdS*

$$\sqrt{\frac{3}{\Lambda}}\frac{\partial}{\partial t}\Psi=\mathrm{i}\mathbf{H}_M\Psi,\quad \mathbf{H}_M:=\mathrm{i}\left(\frac{1+\varrho^2}{2}\right)\gamma^0\left[\gamma^1\frac{\partial}{\partial x^1}+\gamma^2\frac{\partial}{\partial x^2}+\gamma^3\frac{\partial}{\partial x^3}+\frac{2\mathrm{i}M}{1-\varrho^2}\sqrt{\frac{3}{\Lambda}}\right],$$

et vérifiant $\Psi(0)=\Psi_0$. Le potentiel d'interaction gravitationnelle divergeant au bord de $\mathbb{B}$, les conditions aux limites ne sont pas classiques. Quand $M\geqslant\sqrt{\frac{\Lambda}{12}}$, les champs $\Psi_0\in\mathbf{L}^2$ tels que $\mathbf{H}_M\Psi_0\in\mathbf{L}^2$ tendent vers 0 au bord, et le problème de Cauchy admet une solution unique car $\mathbf{H}_M$ est essentiellement auto-adjoint sur $[C_0^\infty(\mathbb{B})]^4$. En revanche, quand $0<M<\sqrt{\frac{\Lambda}{12}}$, les champs $\Psi_0\in\mathbf{L}^2$ avec $\mathbf{H}_M\Psi_0\in\mathbf{L}^2$ peuvent diverger au bord : il existe $\Psi_-\in[H^{1/2}(S^2)]^4$, $\Psi_+\in[L^2(S^2)]^4$, tels que

$$\Psi_0(\varrho\omega)=(1-\varrho)^{-M\sqrt{3/\Lambda}}\Psi_-(\omega)+(1-\varrho)^{M\sqrt{3/\Lambda}}\Psi_+(\omega)+\mathrm{o}_{L^2(S^2_\omega)}\big(\sqrt{1-\varrho}\big),\quad \varrho\to1.$$

De plus Ψ_- et Ψ_+ satisfont respectivement la condition de polarisation *MIT-bag* et la condition *chirale* [4]. Nous construisons des réalisations auto-adjointes de $\mathbf{H}_M$ en imposant une relation linéaire, éventuellement non locale, sur Ψ_+ et Ψ_-. Par exemple, $\Psi_+=0$ (respectivement $\Psi_-=0$) est un analogue asymptotique de la condition *MIT-bag* (respectivement de la condition *chirale*). De même, en notant D_{S^2} l'opérateur de Dirac sur S^2, nous introduisons l'analogue asymptotique de la condition d'Atiyah, Patodi, et Singer,

$$\big\|\mathbf{1}_{]0,\infty[}(D_{S^2})\Psi_0(\varrho\omega)\big\|_{L^2(S^2_\omega)}=\mathrm{o}\big(\sqrt{1-\varrho}\big),\quad \varrho\to1,$$

ce qui revient à imposer $\mathbf{1}_{]0,\infty[}(D_{S^2})\Psi_+=\mathbf{1}_{]0,\infty[}(D_{S^2})\Psi_-=0$. En liant linéairement Ψ_+ et Ψ_-, nous construisons aussi une nouvelle famille de conditions asymptotiques rendant $\mathbf{H}_M$ autoadjoint. Dans tous les cas, le spectre de l'hamiltonien est discret. Ce dernier résultat était attendu à cause de périodicité en temps des géodésiques temporelles de *AdS*. On prouve aussi un résultat d'équipartition de l'énergie : si $\Psi(t)=\mathrm{e}^{\mathrm{i}t\sqrt{\Lambda/3}\,\mathbb{H}}\Psi(0)$ où $\mathbb{H}$ est une réalisation autoadjointe de $\mathbf{H}_M$, $M>0$, alors :

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathbb{B}}\Psi^*\gamma^0\gamma^5\Psi(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t=0,\quad \gamma^5:=-\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3.$$

Les démonstrations s'appuient sur une décomposition du spineur par des harmoniques spinoïdales qui diagonalisent D_{S^2}. L'étude de $\mathbf{H}_M$ est ainsi ramenée à une analyse soigneuse de la famille d'opérateurs sur $L^2(]0,\frac{\pi}{2}[_x)$:

$$\mathrm{i}\gamma^0\gamma^1\frac{\partial}{\partial x}+\frac{l+1/2}{\sin x}\gamma^0\gamma^2-\frac{M}{\cos x}\sqrt{\frac{3}{\Lambda}}\gamma^0,\quad l\in\mathbb{N}+\frac{1}{2}.$$

En particulier, la distinction $M\gtrless\sqrt{\frac{\Lambda}{12}}$ apparaît dans le comportement quand $x\to\frac{\pi}{2}$, des solutions du problème

$$\frac{\mathrm{d}}{\mathrm{d}x}v_\pm\pm\frac{m}{\cos x}v_\pm=f_\pm,\quad v_\pm,f_\pm\in L^2\left(\left]0,\frac{\pi}{2}\right[\right),$$

pour lequel $m=\frac{1}{2}$ est une valeur critique.

1. Dynamics on the Anti-de-Sitter Universe

We consider the Poincaré ball $\mathbb{B}:=\{\mathbf{x}=\varrho\omega=(x^1,x^2,x^3)\in\mathbb{R}^3,\ \varrho\in[0,1[,\omega\in S^2\}$ with its hyperbolic metric $\mathrm{d}s^2_{\mathbb{B}}=\frac{4}{(1-\varrho^2)}(\mathrm{d}\varrho^2+\varrho^2\,\mathrm{d}\omega^2)$. Then, given $\Lambda>0$, the Anti-de-Sitter Universe *AdS* is the Lorentzian manifold defined by

$$AdS=\mathbb{R}_t\times\mathbb{B},\quad \mathrm{d}s^2_{AdS}=\left(\frac{1+\varrho^2}{1-\varrho^2}\right)^2\mathrm{d}t^2-\frac{3}{\Lambda}\,\mathrm{d}s^2_{\mathbb{B}}.$$

Introducing the new radial variable, $x = 2\arctan\varrho$, AdS can be described in spherical coordinates by:

$$AdS = \mathbb{R}_t \times \left[0, \frac{\pi}{2}\right[_x \times [0,\pi]_\theta \times [0,2\pi[_\varphi, \qquad \mathrm{d}s^2_{AdS} = \left(1+\tan^2 x\right)\mathrm{d}s^2_{\mathcal{E}},$$

$$\mathrm{d}s^2_{\mathcal{E}} = \mathrm{d}t^2 - \frac{3}{\Lambda}\left(\mathrm{d}x^2 + \sin^2 x\,\mathrm{d}\theta^2 + \sin^2 x \sin^2\theta\,\mathrm{d}\varphi^2\right).$$

Therefore, if the 3-sphere S^3 is parametrized by $(x,\theta,\varphi) \in [0,\pi]\times[0,\pi]\times[0,2\pi[$, we see that AdS is conformally equivalent to a submanifold $\mathcal{M}$ of the Einstein cylinder ($\mathcal{E} := \mathbb{R}_t \times S^3, \mathrm{d}s^2_{\mathcal{E}}$), and the crucial point is that the boundary $\partial\mathcal{M} = \mathbb{R}_t \times \{x = \frac{\pi}{2}\} \times S^2_{\theta,\varphi}$ is time-like.

As a consequence, AdS is geodesically complete, and time oriented by the Killing vector field ∂_t, but its causality is not at all trivial: (1) given a point P on the slice $t=0$, the future-pointing null geodesics starting from P form a curving cone of which the boundary approaches but does not reach the slice $t = \frac{\pi}{2}\sqrt{\frac{\Lambda}{3}}$, hence AdS is not globally hyperbolic; (2) the future-pointing timelike geodesics being parametrized by $(t,\mathbf{x}(t))_{t\in\mathbb{R}}$, the function $t \mapsto \mathbf{x}(t)$ is t-periodic, with period $\pi\sqrt{\frac{\Lambda}{3}}$. These unusual properties suggest that: (i) we have to add some condition at the 'infinity' $S^2 = \partial\mathbb{B}$ to solve an initial value problem; (ii) the spectrum of the Hamiltonian of the massive fields is discrete. In fact we shall see that (i) crucially depends on the mass of the field and (ii) is indeed true.

The case of the Klein–Gordon equation was considered in [5]; in this Note, we investigate possible dynamics for the spin $\frac{1}{2}$ fields. In Cartesian coordinates, the Dirac equation on AdS has the form:

$$\sqrt{\frac{3}{\Lambda}}\gamma^0\frac{\partial}{\partial t}\Psi + \frac{1+\varrho^2}{2}\left[\gamma^1\frac{\partial}{\partial x^1} + \gamma^2\frac{\partial}{\partial x^2} + \gamma^3\frac{\partial}{\partial x^3} + \frac{2\mathrm{i}M}{1-\varrho^2}\sqrt{\frac{3}{\Lambda}}\right]\Psi = 0, \quad t\in\mathbb{R},\ \mathbf{x}\in\mathbb{B}, \tag{1}$$

where $\gamma^\mu \in \mathbb{C}^{4\times4}$ satisfy $(\gamma^\mu)^* = \eta^{\mu\mu}\gamma^\mu$, $\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\,\mathrm{Id}$, $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Since the charge of the spinor is the formally conserved L^2 norm, we introduce the Hilbert space:

$$\mathbf{L}^2 := \left[L^2\left(\mathbb{B}, \frac{2}{1+\varrho^2}\,\mathrm{d}\mathbf{x}\right)\right]^4,$$

and, given $\Psi_0 \in \mathbf{L}^2$, we want to solve the initial problem, i.e. to find a solution of (1) satisfying:

$$\Psi \in C^0\left(\mathbb{R}_t;\mathbf{L}^2\right), \quad \Psi(t=0,\cdot) = \Psi_0(\cdot). \tag{2}$$

Furthermore, we expect that the solutions satisfy the conservation law:

$$\forall t\in\mathbb{R}, \quad \left\|\Psi(t)\right\|_{\mathbf{L}^2} = \|\Psi_0\|_{\mathbf{L}^2}. \tag{3}$$

Moreover, since ∂_t is a Killing vector field on AdS, it is natural to assume that $t\in\mathbb{R} \mapsto (\Psi_0 \mapsto \Psi(t))$, is a group acting on $\mathbf{L}^2$. Therefore we look for strongly continuous unitary groups $U(t)$ on $\mathbf{L}^2$ that solve (1). According to the Stone theorem, the problem consists in finding self-adjoint realizations on $\mathbf{L}^2$ of the differential operator

$$\mathbf{H}_M := \mathrm{i}\left(\frac{1+\varrho^2}{2}\right)\gamma^0\left[\gamma^1\frac{\partial}{\partial x^1} + \gamma^2\frac{\partial}{\partial x^2} + \gamma^3\frac{\partial}{\partial x^3} + \frac{2\mathrm{i}M}{1-\varrho^2}\sqrt{\frac{3}{\Lambda}}\right],$$

with domain

$$D(\mathbf{H}_M) = \left\{\Psi\in\mathbf{L}^2;\ \mathbf{H}_M\Psi\in\mathbf{L}^2\right\}.$$

2. Self-adjointness

Note that when the mass is null, the Dirac system is conformally invariant, and we could solve the Cauchy problem on the Einstein cylinder $\mathcal{E}$ to easily produce global solutions on AdS; this approach was used by Y. Choquet-Bruhat for the Yang–Mills–Higgs equations [3]. Moreover, when $M=0$, Eq. (1) has smooth coefficients up to the boundary. Therefore, in the case of the zero mass, we deal with a classical mixed hyperbolic problem, and different boundary conditions for the Dirac system with regular potential are well known (see e.g. [1,2,4]), that take the form $\mathcal{B}\Psi(t,\omega) = 0$ for $(t,\omega)\in\mathbb{R}\times S^2$: the *MIT-bag* condition with $\mathcal{B}_{MIT} = \tilde{\gamma}^1 + i\,\mathrm{Id}$, the *Chiral* condition with $\mathcal{B}_{CHI} = \tilde{\gamma}^1 - \mathrm{i}\,\mathrm{Id}$, the non local *APS*-condition introduced by M.F. Atiyah, V.K. Patodi, and I.M. Singer with $\mathcal{B}_{APS} = \mathbf{1}_{]0,\infty[}(D_{S^2})$, and the *mAPS* condition of [4], with $\mathcal{B}_{mAPS} = \mathbf{1}_{]0,\infty[}(D_{S^2})(\tilde{\gamma}^1 + \mathrm{Id})$. Here $\tilde{\gamma}^1 = -\omega_j\gamma^j$ and D_{S^2} is the intrinsic Dirac operator on the two-sphere.

When $M > 0$ the situation is very different because the potential blows up as $\varrho \to 1$. The analogous situation of the infinite mass at the infinity of the Minkowski space has been investigated in [6,7]. In our case, the key result is the asymptotic behaviour, near the boundary, of the spinors of $D(\mathbf{H}_M)$.

Theorem 2.1. *Let Ψ be in $D(\mathbf{H}_M)$ with $M > 0$. Then $\Psi(\varrho\omega) \in [C^0(]0,1[_\varrho; H^{1/2}(S^2_\omega))]^4$.*

When $M > \sqrt{\frac{\Lambda}{12}}$, we have

$$\left\|\Psi(\varrho\omega)\right\|_{L^2(S^2_\omega)} = \mathrm{O}\big(\sqrt{1-\varrho}\big), \quad \varrho \to 1.$$

When $M = \sqrt{\frac{\Lambda}{12}}$, we have

$$\left\|\Psi(\varrho\omega)\right\|_{L^2(S^2_\omega)} = \mathrm{O}\big(\sqrt{(\varrho-1)\ln(1-\varrho)}\big), \quad \varrho \to 1.$$

When $0 < M < \sqrt{\frac{\Lambda}{12}}$, there exists $\Psi_- \in [H^{1/2}(S^2)]^4$, $\Psi_+ \in [L^2(S^2)]^4$, and $\psi \in [C^0([0,1]_\varrho; L^2(S^2_\omega))]^4$ satisfying

$$\Psi(\varrho\omega) = (1-\varrho)^{-m}\Psi_-(\omega) + (1-\varrho)^m\Psi_+(\omega) + \psi(\varrho\omega), \quad m := M\sqrt{\frac{3}{\Lambda}} \in \left]0, \frac{1}{2}\right[, \tag{4}$$

$$\tilde{\gamma}^1\Psi_- + \mathrm{i}\Psi_- = 0, \quad \tilde{\gamma}^1\Psi_+ - \mathrm{i}\Psi_+ = 0, \tag{5}$$

$$\left\|\psi(\varrho\omega)\right\|_{L^2(S^2_\omega)} = \mathrm{o}\big(\sqrt{1-\varrho}\big), \quad \varrho \to 1. \tag{6}$$

Conversely, for any $\Psi_- \in [H^{1/2+m}(S^2)]^4$, $\Psi_+ \in [H^{1/2-m}(S^2)]^4$ satisfying (5)*, there exists $\Psi \in D(\mathbf{H}_M)$ satisfying* (4) *and* (6).

We note that when $M \geqslant \sqrt{\frac{\Lambda}{12}}$, the elements of the domain of $\mathbf{H}_M$ satisfy the homogeneous Dirichlet Condition on $\partial\mathbb{B}$. We shall see that $\mathbf{H}_M$ is self-adjoint. In opposite, when $0 < M < \sqrt{\frac{\Lambda}{12}}$, the trace of Ψ on $\partial\mathbb{B}$ is not defined, the leading term $(1-\varrho)^{-m}\Psi_-$ satisfies the *MIT-bag* Condition and the next term $(1-\varrho)^m\Psi_+$ satisfies the *Chiral* Condition. We introduce natural generalizations of the classic boundary conditions in terms of asymptotic behaviours near S^2:

$$\left\|\mathcal{B}\Psi(\varrho\omega)\right\|_{L^2(S^2_\omega)} = \mathrm{o}\big(\sqrt{1-\varrho}\big), \quad \varrho \to 1,$$

and we consider the operators $\mathbb{H}_{\mathcal{B}}$, $\mathcal{B} = \mathcal{B}_{MIT}, \mathcal{B}_{CHI}, \mathcal{B}_{APS}, \mathcal{B}_{mAPS}$, defined as the differential operator $\mathbf{H}_M$ endowed with the domain

$$D(\mathbb{H}_{\mathcal{B}}) := \left\{\Psi \in D(\mathbf{H}_M);\ \left\|\mathcal{B}\Psi(\varrho\omega)\right\|_{L^2(S^2_\omega)} = \mathrm{o}\big(\sqrt{1-\varrho}\big),\ \varrho \to 1\right\}.$$

These asymptotic constraints on Ψ are conditions on $\Psi_\pm$:

$$D(\mathbb{H}_{\mathcal{B}_{MIT}}) = \left\{\Psi \in D(\mathbf{H}_M);\ \Psi_+ = 0\right\}, \qquad D(\mathbb{H}_{\mathcal{B}_{CHI}}) = \left\{\Psi \in D(\mathbf{H}_M);\ \Psi_- = 0\right\},$$
$$D(\mathbb{H}_{\mathcal{B}_{APS}}) = D(\mathbb{H}_{\mathcal{B}_{mAPS}}) = \left\{\Psi \in D(\mathbf{H}_M);\ \mathbf{1}_{]0,\infty[}(D_{S^2})\Psi_+ = \mathbf{1}_{]0,\infty[}(D_{S^2})\Psi_- = 0\right\}.$$

We now construct a large new family of asymptotic conditions, by imposing a linear relation between Ψ_- and Ψ_+. If we denote $\Psi_\pm = {}^t(\psi^1_\pm, \psi^2_\pm, \psi^3_\pm, \psi^4_\pm)$, the constraints of polarization (5) allow us to express $\psi^{3,4}_\pm$ by using $\psi^{1,2}_\pm$:

$$\begin{pmatrix}\psi^3_\pm(\omega)\\ \psi^4_\pm(\omega)\end{pmatrix} = \pm\mathrm{i}\sum_1^3 \omega^j\sigma^j \begin{pmatrix}\psi^1_\pm(\omega)\\ \psi^2_\pm(\omega)\end{pmatrix}.$$

We consider two densely defined self-adjoint operators $(\mathbf{A}^\pm, D(\mathbf{A}^\pm))$ on $L^2(S^2) \times L^2(S^2)$, satisfying

$$D(\mathbf{A}^+) = L^2(S^2) \times L^2(S^2), \qquad D(\mathbf{A}^-) \supset H^{1/2}(S^2) \times H^{1/2}(S^2),$$
$$\mathbf{A}^\pm\big(C^\infty(S^2) \times C^\infty(S^2)\big) \subset H^{1/2\pm m}(S^2) \times H^{1/2\pm m}(S^2).$$

We define the operators $\mathbb{H}_{\mathbf{A}^+}$, $\mathbb{H}_{\mathbf{A}^-}$ as the differential operator $\mathbf{H}_M$ with the respective domains:

$$D(\mathbb{H}_{\mathbf{A}^\pm}) := \left\{ \Psi \in D(\mathbf{H}_M);\ \begin{pmatrix} \psi^1_\mp \\ \psi^2_\mp \end{pmatrix} = \mathbf{A}^\pm \begin{pmatrix} \psi^1_\pm \\ \psi^2_\pm \end{pmatrix} \right\}.$$

For $\mathbf{A}^- = \mathbf{A}^+ = 0$, we obviously have $\mathbb{H}_{\mathbf{A}^-} = \mathbb{H}_{\mathcal{B}_{MIT}}$, $\mathbb{H}_{\mathbf{A}^+} = \mathbb{H}_{\mathcal{B}_{CHI}}$.

We now state the main theorem of this Note:

Theorem 2.2.

(i) *When* $M \geqslant \sqrt{\frac{\Lambda}{12}}$, $\mathbf{H}_M$ *is essentially self-adjoint on* $[C_0^\infty(\mathbb{B})]^4$.

(ii) *When* $0 < M < \sqrt{\frac{\Lambda}{12}}$, $\mathbb{H}_{\mathbf{A}^+}$, $\mathbb{H}_{\mathbf{A}^-}$, $\mathbb{H}_{\mathcal{B}_{APS}} = \mathbb{H}_{\mathcal{B}_{mAPS}}$ *are self-adjoint on* $\mathbf{L}^2$.

(iii) *The resolvent of any self-adjoint realization of* $\mathbf{H}_M$, $M > 0$, *is compact on* $\mathbf{L}^2$.

We note that the spectrum of these operators is discrete. We plan to study it in a future work. The proofs of these results are made much easier by the use of the spherical coordinates. We introduce a new spinor $\Phi(t, x, \theta, \varphi)$ for $t \in \mathbb{R}$, $x \in [0, \frac{\pi}{2}[$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi[$, by writing:

$$\Psi(t, x^1, x^2, x^3) = \frac{1}{2\tan(x/2)} \mathrm{e}^{\frac{\varphi}{2}\gamma^1\gamma^2} \mathrm{e}^{\frac{\theta}{2}\gamma^3\gamma^1} (I - \gamma^1\gamma^2 - \gamma^2\gamma^3 - \gamma^3\gamma^1)\Phi(t, x, \theta, \varphi).$$

Then Φ is solution of:

$$\sqrt{\frac{3}{\Lambda}}\frac{\partial}{\partial t}\Phi + \gamma^0\gamma^1\frac{\partial}{\partial x}\Phi + \frac{1}{\sin x}\left[\gamma^0\gamma^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\right) + \frac{1}{\sin\theta}\gamma^0\gamma^3\frac{\partial}{\partial\varphi}\right]\Phi + \frac{\mathrm{i}M}{\cos x}\sqrt{\frac{3}{\Lambda}}\gamma^0\Phi = 0.$$

This form of the Dirac equation is interesting because we can diagonalize the angular part by using the spinoidal spherical harmonics. Then the study of $\mathbf{H}_M$ is reduced to a careful analysis of the family of one-dimensional operators on $[L^2(]0, \frac{\pi}{2}[_x)]^4$:

$$\mathrm{i}\gamma^0\gamma^1\frac{\partial}{\partial x} + \frac{l + 1/2}{\sin x}\gamma^0\gamma^2 - \frac{M}{\cos x}\sqrt{\frac{3}{\Lambda}}\gamma^0, \quad l \in \mathbb{N} + \frac{1}{2}.$$

In particular, expansion (4) is related to the behaviour as $x \to \frac{\pi}{2}$ of the solutions of the problem

$$\frac{\mathrm{d}}{\mathrm{d}x}v_\pm \pm \frac{m}{\cos x}v_\pm = f_\pm, \quad v_\pm, f_\pm \in L^2\left(\left]0, \frac{\pi}{2}\right[\right),$$

for which $m = \frac{1}{2}$ is a critical value. We achieve this Note by turning over to the Cauchy problem. We also establish a result of equipartition of the energy:

Theorem 2.3.

(i) *Given* $\Psi_0 \in \mathbf{L}^2$, *there exist solutions of* (1)–(3), *and all the solutions of* (1), (2), *are equal for*

$$(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{B}, \quad |t| < \sqrt{\frac{3}{\Lambda}}\left(\frac{\pi}{2} - 2\arctan\varrho\right). \tag{7}$$

(ii) *When* $M \geqslant \sqrt{\frac{\Lambda}{12}}$, *the Cauchy problem* (1), (2) *has a unique solution. This solution satisfies* (3).

(iii) *Let* $\Psi \in C^0(\mathbb{R}_t; \mathbf{L}^2)$ *be a solution of* (1), *given by* $\Psi(t) = \mathrm{e}^{\mathrm{i}t\sqrt{\frac{\Lambda}{3}}\mathbb{H}}\Psi(0)$ *where* $\mathbb{H}$ *is a self-adjoint realization of* $\mathbf{H}_M$, $M > 0$. *Then we have*:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathbb{B}}\Psi^*\gamma^0\gamma^5\Psi(t, \mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0, \quad \gamma^5 := -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3.$$

It would be interesting to study the role of the asymptotic conditions for the propagation of the singularities and the energy beyond the maximal domain of global hyperbolicity (7).

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