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Partial Differential Equations

Unconditional well-posedness for subcritical NLS in H^s

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Abstract

Let $n \geqslant 3$ and consider the subcritical nonlinear Schrödinger equation, $i\partial_t u + \Delta u = |u|^\alpha u$, with initial data $u_0 \in H^s(\mathbb{R}^n)$. When $s \geqslant 1$, Kato proved that if a maximal solution exists, then it is unique in $C([0, T_{\max}), H^s)$. Previously, uniqueness had only been proven in strictly smaller subspaces. The existence of a solution is assured when $s \in [0, 1]$, so that the subcritical nonlinear Schrödinger equation is unconditionally locally well-posed in H^1. We extend the uniqueness result so that the subcritical nonlinear Schrödinger equation is unconditionally locally well-posed in H^s when $s \in [\frac{n}{2(n-1)}, 1]$. ***To cite this article: K.M. Rogers, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Unicité inconditionnelle pour l'équation de Schrödinger non-linéaire sous-critique dans H^s. On considère l'équation de Schrödinger linéaire sous-critique $i\partial_t u + \Delta u = |u|^\alpha u$, sur $\mathbb{R}^n$, $n \geqslant 3$, à donnée initiale u_0 dans $H^s(\mathbb{R}^n)$. Si $s \geqslant 1$, Kato a démontré que si il existe une solution maximale, elle est unique dans $C([0, T_{\max}), H^s)$. Les seuls résultats d'unicité connus auparavant étaient dans des sous-espaces stricts de cet espace. L'existence d'une solution étant connue pour $s \in [0, 1]$, l'équation de Schrödinger sous-critique est localement bien posée dans H^1 sans condition supplémentaire pour l'unicité. Dans cette Note, nous généralisons le résultat d'unicité de Kato, montrant que l'équation est bien posée avec unicité inconditionnelle dans tous les espaces H^s, $s \in [\frac{n}{2(n-1)}, 1]$. ***Pour citer cet article : K.M. Rogers, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

Let $n \geqslant 3$, $\alpha > 0$ and $s \in [0, \frac{n}{2})$, and consider the Cauchy problem for the nonlinear Schrödinger equation,

$$(\mathrm{NLS}_{s,\alpha}) \qquad \begin{cases} i\partial_t u + \Delta u = \pm|u|^\alpha u, \\ u(\cdot, 0) = u_0 \in H^s(\mathbb{R}^n). \end{cases}$$

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We consider α which are strictly smaller than the critical power under scaling $\alpha_s = \frac{4}{n-2s}$. We denote the smallest integer less than or equal to s by $[s]$. T. Cazenave and F.B. Weissler [2] (see also [6,7]) proved that $\mathrm{NLS}_{s,\alpha}$ is locally well-posed in

$$C\big([0, T_{\max}), H^s(\mathbb{R}^n)\big) \cap L^q_{\mathrm{loc}}\big([0, T_{\max}), B^s_{r,2}(\mathbb{R}^n)\big)$$

for all $\alpha \in ([s], \alpha_s]$, where $q = \frac{4(\alpha+2)}{\alpha(n-2s)}$, $r = \frac{n(\alpha+2)}{n+\alpha s}$, and $B^s_{r,2}$ is the Besov space. The result could be described as 'conditional' in the sense that we restrict attention to an auxiliary space $L^q_{\mathrm{loc}}([0, T_{\max}), B^s_{r,2})$ in order to be sure that the solution is unique.

To prove well-posedness in the unrestricted space $C([0, T_{\max}), H^s)$, it is necessary to prove uniqueness there. This is known as unconditional well-posedness.

Kato [7] proved uniqueness when $\alpha < \min(\frac{2+2s}{n-2s}, \alpha_s)$, so that, combining this with the existence and continuity results, we see that $\mathrm{NLS}_{s,\alpha}$ is unconditionally locally well-posed in the range $\alpha \in ([s], \min(\frac{2+2s}{n-2s}, \alpha_s))$. The restriction $\alpha > [s]$ can be removed when $s = 1$ or $s \in 2\mathbb{N}$, so that subcritical $\mathrm{NLS}_{\alpha,s}$ is unconditionally locally well-posed in these classical Sobolev spaces.

G. Furioli, F. Planchon, and E. Terraneo [4,5] used paraproduct techniques to extend Kato's result. They proved uniqueness in the slightly larger space $C([0, T_{\max}), \dot{H}^s)$ when

$$\max\left(1, s, \frac{2s}{n-2s}\right) < \alpha < \min\left(\frac{2+4s}{n-2s}, \frac{n+2-2s}{n-2s}, \frac{n+2s}{n-2s}, \alpha_s\right). \tag{1}$$

In particular, combining this with the existence and continuity results, 3-dimensional $\mathrm{NLS}_{2,s}$ is unconditionally locally well-posed when $s \in (\frac{1}{2}, 1)$.

For a more thorough account of unconditional well-posedness, including the results in lower dimensions, see [5].

As Kato's uniqueness result is complete in the subcritical range when $s \geqslant 1$, we restrict our attention to the range $s \in [0, 1)$. Here the condition $\alpha > [s]$ is vacuous, so that the subcritical existence theory is complete. By proving uniqueness, we obtain

Theorem 1.1. *Let* $s \in [0, 1]$. *Then* $\mathrm{NLS}_{s,\alpha}$ *is unconditionally locally well-posed in the range* $\frac{2+2s}{n-2s} \leqslant \alpha < \min(\frac{2+4s-4s/n}{n-2s}, \alpha_s)$.

In particular, when $s > \frac{n(n-2)}{6n-4}$, we fill the gap $\frac{2+2s}{n-2s} \leqslant \alpha \leqslant \max(1, s, \frac{2s}{n-2s})$ which was not dealt with by the result of Kato or that of Furioli, Planchon and Terraneo. We will see that the condition $s > \frac{n(n-2)}{6n-4}$ is not overly restrictive in low dimensions.

When $n = 3$ and $s < 1/2$, the condition $\alpha < \frac{2+4s}{n-2s}$ in (1) is the most restrictive, and when $s > 1/2$, the most restrictive condition is $\alpha < \frac{n+2-2s}{n-2s}$. When $s = 1/2$, they both coincide with $\alpha < \alpha_s$, so that subcritical $\mathrm{NLS}_{1/2,\alpha}$ is unconditionally locally well-posed.

On the other hand, combining Theorem 1.1 directly with the result of Kato, we see that we have uniqueness when $\alpha < \min(\frac{2+4s-4s/n}{n-2s}, \alpha_s)$. The condition $\alpha < \alpha_s$ is the most restrictive when $s \geqslant 3/4$, giving

Corollary 1. *Let* $n = 3$ *and* $s = \frac{1}{2}$ *or* $s \in [\frac{3}{4}, 1]$. *Then subcritical* $\mathrm{NLS}_{s,\alpha}$ *is unconditionally locally well-posed.*

When $n = 4$ and $s \geqslant 1/2$, we fill the gap, and the most restrictive condition is $\alpha < \alpha_s$. Thus, we obtain

Corollary 2. *Let* $n = 4$ *and* $s \in [\frac{1}{2}, 1]$. *Then subcritical* $\mathrm{NLS}_{s,\alpha}$ *is unconditionally locally well-posed.*

When $n = 5$, the condition $s > \frac{n(n-2)}{6n-4}$ becomes the most restrictive, giving

Corollary 3. *Let* $n = 5$ *and* $s \in (\frac{15}{26}, 1]$. *Then subcritical* $\mathrm{NLS}_{s,\alpha}$ *is unconditionally locally well-posed.*

Finally, when $n \geqslant 6$ the condition $s > \frac{n(n-2)}{6n-4}$ becomes too restrictive to fill the gap. On the other hand, combining Theorem 1.1 directly with the uniqueness result of Kato, we obtain

Corollary 4. *Let $n \geqslant 6$ and $s \in [\frac{n}{2(n-1)}, 1]$. Then subcritical* $\mathrm{NLS}_{s,\alpha}$ *is unconditionally locally well-posed.*

The proof combines ideas of Kato with an inhomogeneous Strichartz estimate due to M.C. Vilela [10].

2. Proof of Theorem 1.1

By Duhamel's formula, the solution to $\mathrm{NLS}_{s,\alpha}$ can be written as

$$u(x,t) = \mathrm{e}^{\mathrm{i}t\Delta}u_0(x) \mp \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)\Delta}|u|^\alpha u(\cdot,\tau)(x)\,\mathrm{d}\tau,$$

where $\mathrm{e}^{\mathrm{i}t\Delta} f$ denotes the solution to the free problem $\mathrm{i}\partial_t u + \Delta u = 0$, with initial datum $u(\cdot,0) = f$.

The following theorem extends results of R.S. Strichartz [9], J. Ginibre and G. Velo [6], K. Yajima [11], Cazenave and Weissler [1,3], and M. Keel and T. Tao [8]. These type of estimates are closely related to the restriction of the Fourier transform, and the first results in this direction were due to E.M. Stein and P. Tomas.

Theorem 2.1. *(See [10].) Let $r \in [1, \frac{2n}{n+2}]$, $\frac{1}{q} = 1 - \frac{n}{2}(\frac{1}{r} - \frac{1}{p})$, and , $\frac{1}{r} - \frac{2}{n} < \frac{1}{p} \leqslant \frac{n}{n-2}(1 - \frac{1}{r})$. Then*

$$\left\| \int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)\Delta} F(\cdot,\tau)(x)\,\mathrm{d}\tau \right\|_{L_t^\infty([0,T],L_x^p(\mathbb{R}^n))} \leqslant C\|F\|_{L_t^q([0,T],L_x^r(\mathbb{R}^n))}.$$

In order to prove uniqueness, we consider two maximal solutions u and $v \in C([0,T_0],H^s)$, where $T_0 < T_{\max}$. By Sobolev embedding, $H^s(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, where $\frac{1}{p} = \frac{1}{2} - \frac{s}{n}$, so that $\|u - v\|_{L_t^\infty([0,T_0],L^p)}$ is well-defined and finite. We will prove that

$$\|u - v\|_{L_t^\infty([0,T],L_x^p)} \leqslant c\|u - v\|_{L_t^\infty([0,T],L_x^p)},$$

for some $T \in (0,T_0]$ and $c \in (0,1)$, so that $u(t) = v(t)$ in $[0,T]$.

By Duhamel's formula and Theorem 2.1,

$$\begin{aligned}\|u - v\|_{L_t^\infty([0,T],L^p)} &= \left\| \int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)\Delta}\big(|u|^\alpha u - |v|^\alpha v\big)(\cdot,\tau)(x)\,\mathrm{d}\tau \right\|_{L_t^\infty([0,T],L^p)} \\ &\leqslant C\big\||u|^\alpha u - |v|^\alpha v\big\|_{L_t^q([0,T],L_x^r)},\end{aligned}$$

where r, p and q satisfy the conditions of Theorem 2.1. Now it is easy to calculate that

$$\big||u|^\alpha u - |v|^\alpha v\big| \leqslant C|u - v|\big(|u|^\alpha + |v|^\alpha\big),$$

so that by Hölder's inequality,

$$\begin{aligned}\|u - v\|_{L_t^\infty([0,T],L^p)} &\leqslant C\|u - v\|_{L_t^{\frac{qp}{r}} L_x^p} \big\||u|^\alpha + |v|^\alpha\big\|_{L_t^{q\tilde{p}} L_x^{r\tilde{p}}} \\ &\leqslant C\|u - v\|_{L_t^{\frac{qp}{r}} L_x^p}\big(\|u\|^\alpha_{L_t^{q\tilde{p}\alpha} L_x^{r\tilde{p}\alpha}} + \|v\|^\alpha_{L_t^{q\tilde{p}\alpha} L_x^{r\tilde{p}\alpha}}\big),\end{aligned}$$

where $\frac{r}{p} + \frac{1}{\tilde{p}} = 1$.

One can check that by the hypothesis, $\frac{n+2}{n-2s} \leqslant \alpha + 1 \leqslant \frac{2n}{n-2s}$, so that letting $r = \frac{2n}{(\alpha+1)(n-2s)}$, we have that $r \in [1, \frac{2n}{n+2}]$ and $r\tilde{p}\alpha = p$. Now the restriction $\alpha < \alpha_s$ of the hypothesis corresponds to the condition $\frac{1}{r} - \frac{2}{n} < \frac{1}{p}$ that appears in Theorem 2.1, and $\alpha \leqslant \frac{2+4s-4s/n}{n-2s}$ corresponds to the condition $\frac{1}{p} \leqslant \frac{n}{n-2}(1 - \frac{1}{r})$. It can also be calculated that $\frac{qp}{r}$ and $q\tilde{p}\alpha$ are finite, so that by Hölder,

$$\|u - v\|_{L_t^\infty([0,T],L^p)} \leqslant CT^\beta\|u - v\|_{L_t^\infty([0,T],L_x^p)}\big(\|u\|^\alpha_{L_t^\infty([0,T],L_x^p)} + \|v\|^\alpha_{L_t^\infty([0,T],L_x^p)}\big)$$

for some $\beta > 0$. Choosing T sufficiently small,

$$CT^{\beta}\left(\|u\|^{\alpha}_{L_t^\infty([0,T_0],L_x^p)} + \|v\|^{\alpha}_{L_t^\infty([0,T_0],L_x^p)}\right) \leqslant c < 1,$$

so that

$$\|u - v\|_{L_t^\infty([0,T],L_x^p(\mathbb{R}^n))} \leqslant c\|u - v\|_{L_t^\infty([0,T],L_x^p(\mathbb{R}^n))},$$

as required.

We note that the choice of T depended only on $\|u\|^{\alpha}_{L_t^\infty([0,T_0],L_x^p)}$ and $\|v\|^{\alpha}_{L_t^\infty([0,T_0],L_x^p)}$, so that we can iterate the process, considering $u(T) = v(T)$ to be the new initial datum, in order to reach T_0. As we can choose T_0 to be arbitrarily close to $T_{\max}$, we see that the solutions are equal for all time $t \in [0, T_{\max})$, and we are done. □

By a counterexample of Vilela [10], the estimate in Theorem 2.1 cannot hold when $\frac{1}{p} + \frac{1}{r} > 1 + \frac{1}{n}$. Thus, an extension of Theorem 2.1, combined with the previous argument, could only weaken the restriction in Theorem 1.1 to $\alpha < \min(\frac{2+4s}{n-2s}, \alpha_s)$. This would correspond to unconditional well-posedness for subcritical $\mathrm{NLS}_{s,\alpha}$ in the range $s \in [\frac{1}{2}, 1]$. In particular, subcritical unconditional well-posedness in L^2 is well beyond the capabilities of these arguments.

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