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Partial Differential Equations

Concerning the Ladyzhenskaya–Smagorinsky turbulence model of the Navier–Stokes equations

Hugo Beirão da Veiga

Dipartimento di Matematica Applicata "U.Dini", Via Buonarrotti, 1/C, 56127 Pisa, Italy

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Abstract

In some recent papers (see below) we prove regularity results in $L^q(\Omega)$ spaces for the second order derivatives of the velocity and the first order derivatives of the pressure for solutions to Stokes and Navier–Stokes systems of equations with shear thickening viscosity. We take into account only regularity results that hold *up to the boundary*. In the above references we consider flat boundaries. More recently, we have extended the above results to the case of curvilinear boundaries. The aim of this note is to describe these last results, together with suitable comments. ***To cite this article: H. Beirão da Veiga, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Sur le modèle de turbulence de Ladyzhenskaya–Smagorinsky des équations de Navier–Stokes. Dans des articles récents (voir ci-après) nous avons démontré des résultats de régularité dans des espaces L^q pour les dérivées secondes de la vitesse et les dérivées premières de la presssion, pour des systèmes de Stokes et Navier–Stokes avec des viscosités qui dépendent de la partie symétrique du gradient des vitesses. Nous considérons seulement des résultats de régularité valables *jusqu'au bord*. Dans ces articles nous avons considéré des frontières plates. Tous récemment nous avons généraliser ces résultats aux cas des frontières arbitraires. Le but de cette note est de décrire ces résultats, avec des commentaires adéquats. ***Pour citer cet article : H. Beirão da Veiga, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

In the 1960s Olga Ladyzhenskaya proposed the following system of equations

$$\partial_t u + (u \cdot \nabla)u - \nabla \cdot T(u, \pi) = f, \quad \nabla \cdot u = 0, \tag{1}$$

in $\Omega \times]0, T]$, as a model for turbulence phenomena. In this model the stress tensor $\mathbb{T} = -\pi I + \nu_T(u)\mathcal{D}u$ depends on the symmetric part of the gradient of the velocity $\mathcal{D}u = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$, in a polynomial way, with p-rate of growth, for $p > 2$. To avoid additional technical difficulties, we consider the typical model of stress tensor $\mathbb{T}$ given by

E-mail address: bveiga@dma.unipi.it.

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$\nu_T(u) = \nu_0 + \nu_1 |\mathcal{D}u|^{p-2}$, where ν_0 and ν_1 are strictly positive constants. In this case the Stokes Principle (see [16], and [14] page 231) is satisfied.

The first mathematical studies on the above class of equations go back to O.A. Ladyzhenskaya. See [6–10]. The case $p = 3$ was introduced by Smagorinsky, see [15], as a turbulence model. In reference [11], Chap. 2, n. 5, J.-L. Lions considers the case in which $\mathcal{D}u$ is replaced by ∇u. However, in this case the Stokes principle is not satisfied.

Such kind of models were intensively studied in the eighties and nineties by J. Nečas and his school, see [12], in order to study certain particular kinds of fluids, and in particular to describe shear thickening ($p > 2$) and shear thinning ($p < 2$) phenomena. In particular, L^q regularity results, up to the boundary, for $p > 2$ and under the non-slip boundary condition

$$u_{|\Gamma} = 0, \tag{2}$$

are stated in [13]. The mathematical analysis concerning such kind of models is far from being trivial, and usually involves a large amount of delicate arguments of both technical and substantial nature.

In recent papers we have considered the question of regularity up the boundary of solutions to the above kind of modified Navier–Stokes equations. In particular, in [1] and [2], the case $p > 2$ is studied. In these references the very basic results are those proved for the generalized *Stokes stationary problem*

$$\begin{cases} -\dfrac{\nu_0}{2}\Delta u - \nu_1 \nabla \cdot (|\mathcal{D}u|^{p-2}\mathcal{D}u) + \nabla\pi = f, \\ \nabla \cdot u = 0, \end{cases} \tag{3}$$

since the results remain to be true in the presence of the convective term (for $p > 2$, its role is not so crucial), and suitable extensions to the evolution problem can be proved in a quite simple way.

The main aim of the article [4] is to extend our previous results to the case of curvilinear boundaries. This is a quite difficult technical problem, especially for equations containing viscosity depending on the module of the symmetric part of the gradient. For it, the known scheme developed for the case in which coefficients in the equations depends on the module of the gradient (Lions model), does not work (see the Remark 1 below). Our proof is done via a very careful analysis up to boundary, and a suitable application of a modified difference quotient method (this is the relevant novelty of the paper) overcoming the simultaneous appearance of three difficulties: boundary regularity (that is, how to recover the vertical derivatives of $\mathcal{D}u$ from the tangential ones), the divergence constraint to be met at each choice of the test functions, and the fact that the system actually depends on the symmetric part of the gradient, rather than on the gradient itself. This leads to the introduction of a certain number of interesting new tricks. The results are anyway proved by first arguing locally, via a suitable flattening of the boundary, and then by a covering argument to recover the final global estimate.

2. The stationary problem

Below Ω is a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary $\Gamma = \partial\Omega$, a manifold of class C^2. For saving space, in this note we do not write estimates. Concerning the stationary case, we prove three main theorems. Theorem 2.1, which can be considered as a 'starting point', ensures the global higher differentiability of the solution (u, π). Rather explicit a priori estimates are also proved.

Theorem 2.1. *Let be $2 \leqslant p \leqslant 3$. Assume that $f \in L^2(\Omega)$ and let (u, π) be the weak solution to problem* (3) *under the boundary condition* (2). *Assume, in addition, that $\mathcal{D}u \in L^q(\Omega)$ for some $p \leqslant q \leqslant 6$. Then*

$$u \in W^{2,r}(\Omega), \quad \nabla\pi \in L^{\bar{p}}(\Omega), \quad \textit{and} \quad \pi \in L^r(\Omega), \tag{4}$$

where $1/r = (p-2)/2q + 1/2$ and $1/\bar{p} = (p-2)/q + 1/2$.

Note that at this stage the assumption $\mathcal{D}u \in L^q$ is only satisfied with $q = p$, since a solution, by standard monotonicity methods, can be initially found in $W^{1,p}$. Therefore the application of this result with $q = p$ leads to the following regularity result:

Theorem 2.2. *Let be $2 \leqslant p \leqslant 3$. Assume that $f \in L^2(\Omega)$ and let (u,π) be the weak solution to problem* (3) *under the boundary condition* (2). *Then*

$$u \in W^{2,p'}(\Omega), \quad \nabla\pi \in L^{p_0}(\Omega), \quad \textit{and} \quad \pi \in L^{p'}(\Omega), \tag{5}$$

where $p_0 = 2p/(3p-4)$.

The next step, which leads to the main regularity result of the paper, is to iterate Theorem 2.1 starting from Theorem 2.2 as follows: Theorem 2.2 allows to get higher integrability of $\mathcal{D}u$ via standard Sobolev embedding, let's say $\mathcal{D}u \in L^q$; then Theorem 2.1 allows to get higher integrability of second derivatives; in turn this implies higher integrability of $\mathcal{D}u$ and so on. Note that this kind of iterations usually lead to establish higher integrability for every exponent strictly less than a certain limiting one, say l. Actually, the exponent l can be reached since in Theorems 2.1 and 2.2 explicit boundary estimates are provided, in turn allowing for a precise control of the constants in the iteration procedure. We have therefore the following result:

Theorem 2.3. *Let be $2 \leqslant p \leqslant 3$, and let f, u and π be as in Theorem 2.2. Then*

$$u \in W^{2,l}(\Omega) \quad \textit{and} \quad \nabla\pi \in L^m(\Omega), \tag{6}$$

where $l = 3(4-p)/(5-p)$ and $m = 6(4-p)/(8-p)$.

Denote by s^* the immersion Sobolev exponent for which $W^{1,s} \subset L^{s^*}$. In Theorem 2.3 the exponent l turns out to be just the exponent for which Theorem 2.1, with $q = l^*$, yields $u \in W^{2,l}$. Then, by a Sobolev embedding theorem, $u \in W^{1,l^*}$. In other words, $q = l^*$ is the fixed point of the map $q \to r \to r^*$. So, further regularity cannot be obtained by appealing to Theorem 2.1.

The extension to solutions to the stationary generalized Navier–Stokes system (obtained by addition of $(u\cdot\nabla)u$ to the left hand side of (3)) is straightforward.

Theorem 2.4. *All the regularity results stated in the Theorems* 2.1*,* 2.2 *and* 2.3 *hold for the generalized Navier–Stokes equations.*

Finally, we turn to examine the situation in the evolution case, where similar results can be obtained provided the exponent p is large enough to avoid the appearance of super-critical non-linearities in the convective term.

3. The evolution problem

The regularity up to the boundary of the second order derivatives in the time-dependent case

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu_0\nabla\cdot\mathcal{D}u - \nu_1\nabla\cdot(|\mathcal{D}u|^{p-2}\mathcal{D}u) + \nabla\pi = f, \\ \nabla\cdot u = 0, \\ u(0) = u_0(x), \end{cases} \tag{7}$$

is easily reduced to a steady state problem in which the solution depends on t as a parameter, and the derivative in t is considered as part of the right-hand side. This is why (in the presence of the convective term) an additional restriction on p comes up: $p \geqslant 2 + 2/5$. This restriction provides solutions u such that $\partial_t u \in L^2(0,T;L^2(\Omega))$. Note that $L^2(\Omega)$ is exactly the regularity assumed here for the external force field f. This situation allows us to consider $\partial_t u$ as part of the right-hand side of our main equation, and hence allows us to regularize the solution with respect to the space variables simply by appealing to our regularity theorems for stationary solutions.

We prove the following results:

Theorem 3.1. *Let u be a weak solution to problem* (7) *under the boundary condition* (2)*, where $u_0 \in W_0^{1,p}(\Omega)$ is a given divergence free vector field and $f \in L^2(0,T;L^2)$. Assume that $2 + 2/5 \leqslant p \leqslant 3$. Then $u \in L^2(0,T;W^{2,p'}) \cap L^\infty(0,T;W^{1,p})$, $\nabla\pi \in L^2(0,T;L^{p_0})$, and $\partial_t u \in L^2(0,T;L^2)$.*

Theorem 3.2. *Under the assumptions of Theorem* 3.1 *one has* $u \in L^{4-p}(0,T;W^{2,l}) \cap L^{\infty}(0,T;W^{1,p})$ *and* $\nabla\pi \in L^{\frac{2(4-p)}{p}}(0,T;L^m)$.

In both theorems the assumption $2 + 2/5 \leqslant p$ is not necessary if the convective term is not present in the equations.

Remark 1. *On the regularity up to the boundary*. In proving interior regularity by the classical translation method, the translations are admissible in all the n independent directions. This allows suitable L^2-estimates for $\nabla\mathcal{D}u$, where the full gradient ∇ is obtained thanks to the possibility of appealing to translations in all the directions. On the other hand, $c|\nabla\nabla u| \leqslant |\nabla\mathcal{D}u| \leqslant C|\nabla\nabla u|$. These two facts together lead to a small distinction if we replace $\mathcal{D}u$ by ∇u in the expression of the stress tensor. However, in proving regularity up to the boundary, the two cases are completely distinct. In fact, solutions to the J.-L. Lions model belong to $W^{2,2}$ up to the boundary. It seems not accidental that there is a very extensive literature on interior regularity for the above class of problems but, as far as we know, scant literature concerning regularity up to the boundary, in the 3D case.

Remark 2. *The case* $p < 2$. The main lines of the method followed here can be adapted to the case $p < 2$. In reference [3] we consider the equation (3) with the term $|\mathcal{D}u|^{p-2}$ replaced by $(1 + |\mathcal{D}u|)^{p-2}$, $p < 2$, and with $\nu_0 = 0$. This last assumption requires a significant improvement of our approach. This improvement diminishes the gap between the Ladyzhenskaya and the Lions models in the treatment of boundary value problems (the case in which the distinction is relevant). Extension to curvilinear boundaries of the results in [3] is shown in the forthcoming paper [5].

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