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Mathematical Analysis

Microlocalization of subanalytic sheaves

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Abstract

In this Note we define specialization and microlocalization for sheaves on the subanalytic site. Applying these functors to the sheaves of tempered and Whitney holomorphic functions we get a unifying description of tempered and formal microlocalization of Andronikof (1994) and Colin (1998). ***To cite this article: L. Prelli, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Microlocalisation des faisceaux sous-analytiques. Dans cette Note, nous définissons la spécialisation et la microlocalisation des faisceaux sous-analytiques. En appliquant ces foncteurs aux faisceaux des fonctions holomorphes tempérées et de Whitney on obtient une description unifiée de la microlocalisation tempérée et formelle de Andronikof (1994) et de Colin (1998). ***Pour citer cet article : L. Prelli, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Soit X une variété analytique réelle et soit k un champ. Nous considérons le site X_{sa} associé à la famille des ouverts sous-analytiques, introduit dans [5], la catégorie des faisceaux sur ce site, et sa catégorie dérivée $D^b(k_{X_{sa}})$. Dans cette catégorie on peut définir les six opérations de Grothendieck et on obtient les formules usuelles (formule de projection, de changement de base, de Künneth, etc.).

Soit E un fibré vectoriel sur une variété analytique réelle Z, et soit E^* son dual. Un ouvert U de E_{sa} est dit $\mathbb{R}^+$-connexe si ses intersections avec les orbites de l'action de $\mathbb{R}^+$ sont connexes et on appelle $\mathbb{R}^+U$ l'ouvert conique associé. Nous définissons la catégorie des faisceaux coniques, i.e. les faisceaux sous-analytiques F satisfaisant $\Gamma(U;F) \simeq \Gamma(\mathbb{R}^+U;F)$ pour tout ouvert $\mathbb{R}^+$-connexe relativement compact U de E_{sa}. Il s'agit d'une définition différente de la définition classique (faisceau équivariant par l'action de $\mathbb{R}^+$). Nous considérons la catégorie dérivée des faisceaux coniques $D^b_{\mathbb{R}^+}(k_{E_{sa}})$. En adaptant les constructions classiques de [3], nous définissons la transformée de Fourier–Sato et la transformée inverse de Fourier–Sato :

$$(\cdot)^\wedge : D^b_{\mathbb{R}^+}(k_{E_{sa}}) \to D^b_{\mathbb{R}^+}(k_{E^*_{sa}}), \qquad F^\wedge = Rp_{2!!}(p_1^{-1}F)_{P'},$$

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$$(\cdot)^\vee : D^b_{\mathbb{R}^+}(k_{E^*_{sa}}) \to D^b_{\mathbb{R}^+}(k_{E_{sa}}), \qquad F^\vee = Rp_{1*}\mathrm{R}\Gamma_{P'}p_2^! F,$$

où $P' := \{(x,y) \in E\times_Z E^*;\ \langle x,y\rangle \leqslant 0\}$, les applications p_1, p_2 sont les projections de $E\times_Z E^*$ sur E et sur E^* respectivement et $p_{2!!}$ désigne l'image directe propre pour les faisceaux sous-analytiques.

Théorème 0.1. *Les foncteurs* $^\wedge$ *et* $^\vee$ *sont des équivalences de catégories inverses l'une de l'autre.*

Soit M une sous-variété analytique d'une variété analytique réelle X et soient $T_M X$ et $T^*_M X$ le fibré normal et le fibré conormal respectivement. Nous considérons la déformation normale $\widetilde{X}_M$ de X le long de M visualisée dans le diagramme (1). Nous définissons les foncteurs de spécialisation et de microlocalisation

$$\nu^{sa}_M : D^b(k_{X_{sa}}) \to D^b_{\mathbb{R}^+}(k_{T_M X_{sa}}), \qquad \nu^{sa}_M(F) = s^{-1}R\Gamma_\Omega p^{-1}F,$$
$$\mu^{sa}_M : D^b(k_{X_{sa}}) \to D^b_{\mathbb{R}^+}(k_{T^*_M X_{sa}}), \qquad \mu^{sa}_M F = (\nu^{sa}_M F)^\wedge.$$

Nous obtenons le triangle distingué de Sato:

$$F|_M \otimes \omega_{M/X} \to \mathrm{R}\Gamma_M F|_M \to R\dot{\pi}_* \mu^{sa}_M F \xrightarrow{+}.$$

Soit Δ la diagonale de $X\times X$, et considérons la déformation normale de $X\times X$ le long de la diagonale (voir le diagramme (2)). Soient $F \in D^b_{\mathbb{R}\text{-c}}(k_X)$ et $G \in D^b(k_{X_{sa}})$. Nous définissons :

$$\mu hom^{sa}(F,G) := \mu^{sa}_\Delta R\mathcal{H}om(q_2^{-1}F, q_1^! G).$$

Soit $T^*X \xrightarrow{\pi} X$ le fibré conormal et considérons la déformation normale de $T^*X \times_X T^*X$ le long de la diagonale. Soit $P_\omega = \{(x,\xi,v_x,v_\xi) \mid \langle v_x,\xi\rangle \geqslant 0\} \subset TT^*X$ et $F \in D^b(k_{X_{sa}})$. Nous établissons aussi une relation avec le foncteur de microlocalisation de [6]

$$\mu F = Rq_{1!!}(K_{T^*X} \otimes q_2^{-1}\pi^{-1}F),$$

où $K_{T^*X} = Rp_{!!}(k_{\overline{\Omega}} \otimes \rho_! k_{P_\omega}) \otimes \rho_! \omega^{\otimes -1}_{\Delta_{T^*X}|T^*X\times T^*X}$.

Théorème 0.2. *Soient* $F \in D^b_{\mathbb{R}\text{-c}}(k_X)$ *et* $G \in D^b(k_{X_{sa}})$.

(i) *Il existe un morphisme naturel*

$$\varphi : R\mathcal{H}om(\pi^{-1}F, \mu G) \to \mu hom^{sa}(F,G).$$

(ii) *Soit* $\rho : T^*X \to T^*X_{sa}$ *le foncteur naturel de sites. Alors* $\rho^{-1}(\varphi)$ *est un isomorphisme.*

Lorsque X est une variété analytique complexe, on considère les faisceaux sous-analytiques $\mathcal{O}^t_X$ et $\mathcal{O}^{\mathrm{w}}_X$ des fonctions holomorphes tempérées et de Whitney respectivement. Nous obtenons une description unifiée de la microlocalisation tempérée et formelle.

Théorème 0.3. *Soit* $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. *Alors on a les isomorphismes*

$$\rho^{-1}\mu hom^{sa}(F, \mathcal{O}^t_X) \simeq t\mu hom(F, \mathcal{O}_X) \quad \text{et} \quad \rho^{-1}\mu hom^{sa}(F, \mathcal{O}^{\mathrm{w}}_X) \simeq \big(D'F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X\big)^a,$$

où $t\mu hom(\cdot, \mathcal{O}_X)$ *est le foncteur de microlocalisation tempérée* [1] *et* $\cdot \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X$ *est le foncteur de microlocalisation formelle* [2].

1. Sheaves on subanalytic sites

The results of Section 1 are extracted from [5].

Let X be a real analytic manifold and let k be a field. Denote by $\mathrm{Op}_{sa}(X)$ the category of subanalytic subsets of X. One endows $\mathrm{Op}_{sa}(X)$ with the following topology: $S \subset \mathrm{Op}_{sa}(X)$ is a covering of $U \in \mathrm{Op}_{sa}(X)$ if for any compact K of X there exists a finite subset $S_0 \subset S$ such that $K \cap \bigcup_{V\in S_0} V = K \cap U$. We will call X_{sa} the subanalytic site.

Let $\mathrm{Mod}(k_{X_{sa}})$ denote the category of sheaves on X_{sa} and let $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ be the Abelian category of $\mathbb{R}$-constructible sheaves on X.

We denote by $\rho : X \to X_{sa}$ the natural morphism of sites. We have functors

$$\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X) \subset \mathrm{Mod}(k_X) \underset{\rho^{-1}}{\overset{\rho_*}{\rightleftarrows}} \mathrm{Mod}(k_{X_{sa}}).$$

The functors ρ^{-1} and ρ_* are the functors of inverse image and direct image associated to ρ. The functor ρ^{-1} admits a left adjoint, denoted by $\rho_!$. The sheaf $\rho_! F$ is the sheaf associated to the presheaf $\mathrm{Op}_{sa}(X) \ni U \mapsto F(\overline{U})$.

The functor ρ_* is fully faithful and exact on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and we identify $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ with its image in $\mathrm{Mod}(k_{X_{sa}})$ by ρ_*.

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a real analytic map. We get the internal operations $\mathcal{H}om$, $\otimes$ and the external operations f^{-1} and f_*, which are always defined for sheaves on Grothendieck topologies. For subanalytic sheaves we can also define the functor of proper direct image $f_{!!}$. The notation $f_{!!}$ is due to the fact that $f_{!!} \circ \rho_* \not\simeq \rho_* \circ f_!$ in general. While the functors f^{-1} and $\otimes$ are exact, the functors $\mathcal{H}om$, f_* and $f_{!!}$ are left exact and admit right derived functors. The functor $Rf_{!!}$ admits a right adjoint, denoted by $f^!$, and we get the usual isomorphisms like projection formula, base change formula, Künneth formula.

2. Fourier–Sato transform

We shall adapt for subanalytic sheaves the classical construction of Fourier–Sato transform for which we refer to [3].

Let E be a vector bundle over a real analytic manifold Z endowed with the natural action μ of $\mathbb{R}^+$, the multiplication on the fibers. Let U be an open subset of E. We say that U is $\mathbb{R}^+$-connected if its intersections with the orbits of μ are connected. We denote $\mathbb{R}^+U$ the conic open set associated to U (i.e. $\mathbb{R}^+U = \mu(U, \mathbb{R}^+)$).

Definition 2.1. A sheaf F on E_{sa} is said conic if $\Gamma(\mathbb{R}^+U; F) \xrightarrow{\sim} \Gamma(U; F)$ for each $\mathbb{R}^+$-connected relatively compact open subanalytic subset U of E. We call $\mathrm{Mod}_{\mathbb{R}^+}(k_{E_{sa}}) \subset \mathrm{Mod}(k_{E_{X_{sa}}})$ the category of conic sheaves and $D^b_{\mathbb{R}^+}(k_{E_{sa}})$ the subcategory of $D^b(k_{E_{sa}})$ consisting of objects with conic cohomology.

Remark that this definition is different from the classical one. Let us consider the diagram

$$E \times \mathbb{R}^+ \underset{p}{\overset{\mu}{\rightrightarrows}} E,$$

where p denotes the projection. One can define the subcategory $\mathrm{Mod}^{\mu}(k_{E_{sa}})$ of $\mathrm{Mod}(k_{E_{sa}})$ consisting of sheaves satisfying $\mu^{-1}F \simeq p^{-1}F$. The categories $\mathrm{Mod}^{\mu}(k_{E_{sa}})$ and $\mathrm{Mod}_{\mathbb{R}^+}(k_{E_{sa}})$ are not equivalent in general. The category of conic subanalytic sheaves has many good properties, for example it is equivalent to the category of sheaves on the conic subanalytic site $E_{sa,\mathbb{R}^+}$ (i.e. the category of open conic subanalytic subsets of E with the topology induced from E_{sa}). This equivalence is strictly related to the geometry of subanalytic subsets, in particular to the fact that any $U \in \mathrm{Op}(E_{sa})$ has a locally finite covering $\{U_i\} \subset \mathrm{Op}(E_{sa})$ consisting of $\mathbb{R}^+$-connected open subsets (i.e. the topology of E_{sa} has a basis consisting of $\mathbb{R}^+$-connected open subsets).

Let E^* be the dual vector bundle and consider the projections p_1, p_2 from $E \times_Z E^*$ to E and E^* respectively. We set $P' := \{(x, y) \in E \times_Z E^*;\ \langle x, y\rangle \leqslant 0\}$. As in classical sheaf theory, we define the Fourier–Sato transform and the inverse Fourier–Sato transform

$$(\cdot)^\wedge : D^b_{\mathbb{R}^+}(k_{E_{sa}}) \to D^b_{\mathbb{R}^+}(k_{E^*_{sa}}), \qquad F^\wedge = Rp_{2!!}(p_1^{-1}F)_{P'},$$
$$(\cdot)^\vee : D^b_{\mathbb{R}^+}(k_{E^*_{sa}}) \to D^b_{\mathbb{R}^+}(k_{E_{sa}}), \qquad F^\vee = Rp_{1*}\mathrm{R}\Gamma_{P'}\, p_2^! F.$$

These constructions are compatible with the classical Fourier–Sato transform of [3], in fact the functors $^\wedge$ and $^\vee$ commute with ρ^{-1} and $R\rho_*$.

Theorem 2.1. *The functors $^\wedge$ and $^\vee$ are equivalence of categories, inverse to each others.*

3. Microlocalization

While the following results are similar to [3], the constructions are delicate and make use of the geometrical properties of subanalytic subsets (or, more generally, of the tools of o-minimal geometry), in particular the cylindrical cell decomposition of a subanalytic subset. Let X be a real n-dimensional analytic manifold and let M be a closed submanifold of codimension ℓ. As usual we denote by $T_M X \tau \to M$ the normal bundle and by $T^*_M X \xrightarrow{\pi} M$ the conormal vector bundle.

We consider the normal deformation of X, i.e. an analytic manifold $\tilde{X}_M$, an application $(p,t): \tilde{X}_M \to X \times \mathbb{R}$, and an action of $\mathbb{R} \setminus \{0\}$ on $\tilde{X}_M$ $(\tilde{x}, r) \mapsto \tilde{x} \cdot r$ such that $p^{-1}(X \setminus M) \simeq (X \setminus M) \times (\mathbb{R} \setminus \{0\})$, $t^{-1}(c) \simeq X$ for each $c \neq 0$ and $t^{-1}(0) \simeq T_M X$. Let $s: T_M X \hookrightarrow \tilde{X}_M$ be the inclusion, Ω the open subset of $\tilde{X}_M$ defined by $\{t > 0\}$, $i_\Omega : \Omega \hookrightarrow \tilde{X}_M$ and $\tilde{p} = p \circ i_\Omega$. We get a commutative diagram

$$\begin{array}{ccccc} T_M X & \xrightarrow{s} & \tilde{X}_M & \xleftarrow{i_\Omega} & \Omega \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \tilde{p}} & \\ M & \xrightarrow{i_M} & X. & & \end{array} \tag{1}$$

Let S be a subset of X. The normal cone to S along M, denoted by $C_M(S)$, is the closed conic subset of $T_M X$ defined by $C_M(S) = T_M X \cap \overline{\tilde{p}^{-1}(S)}$.

Definition 3.1. The specialization along M is the functor

$$\nu^{sa}_M : D^b(k_{X_{sa}}) \to D^b_{\mathbb{R}^+}(k_{T_M X_{sa}}), \quad \nu^{sa}_M(F) = s^{-1} R\Gamma_\Omega p^{-1} F.$$

The microlocalization of F along M is the Fourier–Sato transform of the specialization, i.e.

$$\mu^{sa}_M : D^b(k_{X_{sa}}) \to D^b_{\mathbb{R}^+}(k_{T^*_M X_{sa}}), \quad \mu^{sa}_M F = (\nu^{sa}_M F)^\wedge.$$

We can give a description of the sections of the specialization and the microlocalization of F and we obtain formulas which are similar to the classical ones. For example, let V be a conic subanalytic open subset of $T_M X$. Then:

$$H^j(V; \nu^{sa}_M F) \simeq \varinjlim_U H^j(U; F),$$

where U ranges through the family of $\mathrm{Op}_{sa}(X)$ such that $C_M(X \setminus U) \cap V = \varnothing$. Now let V be a convex conic open subanalytic subset of $T^*_M X$. Then:

$$H^j(V; \mu^{sa}_M F) \simeq \varinjlim_{U,Z} H^j_Z(U; F),$$

where U ranges through the family of $\mathrm{Op}_{sa}(X)$ such that $U \cap M = \pi(V)$, and Z through the family of closed subanalytic subsets such that $C_M(Z) \subset V^\circ$ (V° is the polar cone). Thanks to this formula we can check the compatibility with the classical specialization of [3]: we have the isomorphism $\rho^{-1} \nu^{sa}_M R\rho_* \simeq \nu_M$.

Note that $\rho^{-1} \nu^{sa}_M \not\simeq \nu_M \rho^{-1}$ in general. For example let $X = \mathbb{C}$ and consider the subanalytic sheaf $\mathcal{O}^t_{\mathbb{C}}$ of tempered holomorphic functions defined in Section 4. While $\nu_{\{0\}} \rho^{-1} \mathcal{O}^t_{\mathbb{C}} \simeq \nu_{\{0\}} \mathcal{O}_{\mathbb{C}}$, outside the origin of $\mathbb{C} \simeq T_0 \mathbb{C}$ the sheaf $\rho^{-1} \nu^{sa}_{\{0\}} \mathcal{O}^t_{\mathbb{C}}$ is the sheaf $\mathcal{A}^{\leqslant 0}$ of holomorphic functions with moderate growth at the origin of [7]. In a similar way we can define functorially the sheaves $\mathcal{A}^{<0}$ and $\mathcal{A}$ of [7] using the sheaf of Whitney holomorphic functions $\mathcal{O}^{\mathrm{W}}_{\mathbb{C}}$.

We get the Sato's triangle for subanalytic sheaves:

$$F|_M \otimes \omega_{M/X} \to \mathrm{R}\Gamma_M F|_M \to R\dot{\pi}_* \mu^{sa}_M F \xrightarrow{+}$$

where $\dot{\pi}$ is the restriction of π to $T^*_M X \setminus M$.

Let Δ be the diagonal of $X \times X$, and denote by δ the diagonal embedding. The normal deformation of the diagonal in $X \times X$ can be visualized by the following diagram

$$\begin{array}{ccccccc} TX & \xrightarrow{\sim} & T_\Delta(X \times X) & \xrightarrow{s} & \widetilde{X \times X} & \xleftarrow{i_\Omega} & \Omega \\ & & \downarrow{\scriptstyle \tau_X} & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \tilde{p}} & \\ & & \Delta & \xrightarrow{\delta} & X \times X & \overset{q_2}{\underset{q_1}{\rightrightarrows}} & X. \end{array} \tag{2}$$

Set $p_i = q_i \circ p$, $i = 1, 2$. While $\tilde{p}$ and p_i, $i = 1, 2$, are smooth, p is not, and moreover the square is not Cartesian.

Definition 3.2. Let $F \in D^b_{\mathbb{R}\text{-c}}(k_X)$ and $G \in D^b(k_{X_{sa}})$. We set $\mu hom^{sa}(F, G) := \mu^{sa}_\Delta R\mathcal{H}om(q_2^{-1}F, q_1^! G)$.

Let us study the relation with the microlocalization functor of [6]. Let us consider the cotangent bundle $T^*X \xrightarrow{\pi} X$ and the normal deformation of the diagonal in $T^*X \times T^*X$. Set $P_\omega = \{(x, \xi, v_x, v_\xi) \mid \langle v_x, \xi\rangle \geqslant 0\} \subset TT^*X$ and let $F \in D^b(k_{X_{sa}})$. In [6] one defines

$$\mu F = Rq_{1!!}(K_{T^*X} \otimes q_2^{-1}\pi^{-1}F),$$

where $K_{T^*X} = Rp_{!!}(k_{\overline{\Omega}} \otimes \rho_! k_{P_\omega}) \otimes \rho_! \omega^{\otimes -1}_{\Delta_{T^*X}|T^*X \times T^*X}$.

Theorem 3.3. *Let $F \in D^b_{\mathbb{R}\text{-c}}(k_X)$ and $G \in D^b(k_{X_{sa}})$. There is a natural morphism $\varphi : R\mathcal{H}om(\pi^{-1}F, \mu G) \to \mu hom^{sa}(F, G)$ and $\rho^{-1}(\varphi)$ is an isomorphism.*

4. Tempered and formal microlocalization

Let M be a real analytic manifold. One denotes by $\mathcal{D}b_M$ and $\mathcal{C}^\infty_M$ the sheaves of Schwartz's distributions and $\mathcal{C}^\infty$ functions respectively. We recall the definitions of the sheaves of tempered distributions $\mathcal{D}b^t_M$ and Whitney $\mathcal{C}^\infty$ functions $\mathcal{C}^{\infty,\mathrm{w}}_M$ on M_{sa} of [5]. We have:

$$\Gamma(U; \mathcal{D}b^t_M) = \Gamma(M; \mathcal{D}b_M)/\Gamma_{M\setminus U}(M; \mathcal{D}b_M), \qquad \Gamma(U; \mathcal{C}^{\infty,\mathrm{w}}_M) = \Gamma(M; \mathcal{C}^\infty_M)/\Gamma(M; \mathcal{I}^\infty_{M\setminus U}),$$

where U is a locally cohomologically trivial subanalytic subset and $\Gamma(M; \mathcal{I}^\infty_{M\setminus U})$ denotes the space of $\mathcal{C}^\infty$ functions vanishing on $M \setminus U$ with infinite order.

Now let X be a complex manifold, $X_\mathbb{R}$ the underlying real analytic manifold and $\overline{X}$ the complex conjugate manifold. One denotes by $\mathcal{D}_X$ the sheaf of finite order differential operators with holomorphic coefficients, $\mathcal{O}^t_X$ and $\mathcal{O}^\mathrm{w}_X$ the sheaves of tempered and Whitney holomorphic functions respectively. They are defined as follows:

$$\mathcal{O}^t_X := R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X_\mathbb{R}}), \qquad \mathcal{O}^\mathrm{w}_X := R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\mathrm{w}}_{X_\mathbb{R}}).$$

Let us consider the normal deformation of the diagonal in $X \times X$ as in diagram 2. Let $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. We recall the definitions of Andronikof's tempered microlocalization [1] and Colin's formal microlocalization [2]:

$$t\mu hom(F, \mathcal{O}_X) := \big(s^{-1}\big(\mathcal{D}_{\widetilde{X \times X} \xleftarrow{p_1} X} \otimes_{\mathcal{D}_{\widetilde{X\times X}}} T\mathcal{H}om\big((p_2^{-1}F)_\Omega, \mathcal{O}_{\widetilde{X\times X}}\big)[-1]\big)\big)^\wedge,$$

$$F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X := \big(s^{-1}R\mathcal{H}om\big(\mathcal{D}_{\widetilde{X\times X} \xrightarrow{p_1} X}, (p_2^{-1}F)_{\overline{\Omega}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\widetilde{X\times X}}\big)\big)^\vee.$$

Here $T\mathcal{H}om(\cdot, \mathcal{O}_X)$ and $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ denote the functors of tempered and formal cohomology respectively (see [4] for details).

Theorem 4.1. *Let $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. We have the isomorphisms*

$$\rho^{-1}\mu hom^{sa}(F, \mathcal{O}^t_X) \simeq t\mu hom(F, \mathcal{O}_X) \quad \textit{and} \quad \rho^{-1}\mu hom^{sa}(F, \mathcal{O}^\mathrm{w}_X) \simeq \big(D'F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X\big)^a,$$

where $(\cdot)^a$ denotes the direct image for the antipodal map.

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