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Probability Theory

A nonadapted version of the invariance principle of Peligrad and Utev

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Abstract

We present a nonadapted version of the invariance principle of Peligrad and Utev [M. Peligrad, S. Utev, A new maximal inequality and invariance principle for stationary sequences, Ann. Probab. 33 (2005) 798–815]. ***To cite this article: D. Volný, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Version non adaptée du principe d'invariance de Peligrad et Utev. Nous présentons une version non adaptée du principe d'invariance de Peligrad et Utev [M. Peligrad, S. Utev, A new maximal inequality and invariance principle for stationary sequences, Ann. Probab. 33 (2005) 798–815]. ***Pour citer cet article : D. Volný, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Let $(\Omega, \mathcal{A}, \mu, T)$ be a dynamical system where T is a bijective, bimeasurable and measure preserving map of Ω onto Ω. By U we denote the operator on the space of all measurable functions on Ω defined by $Uf = f \circ T$, $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ is a filtration, $\mathcal{F}_i \subset T^{-1}\mathcal{F}_i = \mathcal{F}_{i+1}$. For a measurable function f we denote $S_n(f) = \sum_{i=0}^{n-1} U^i f$. In [5], Maxwell and Woodroofe proved that if $f \in L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ is $\mathcal{F}_0$-measurable and

$$\sum_{k=1}^{\infty} \frac{\|E(S_k(f)\,|\,\mathcal{F}_0)\|_2}{k^{3/2}} < \infty \tag{1}$$

then there exists a martingale difference sequence $(U^i m)$ (adapted to the filtration $(\mathcal{F}_i)$) approximating $(U^i f)$, i.e.

$$\big\| E\big(S_k(f-m)\big)\big\|_2 = \mathrm{o}(\sqrt{n}), \tag{2}$$

which implies a central limit theorem for $(U^i f)$ (cf. [3]). In [6] Peligrad and Utev proved a new maximal inequality which implies that under (1) we get also the weak invariance principle. In [10] Volný found a method enabling to

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prove a nonadapted version of the Maxwell–Woodroofe's CLT. In the article the martingale approximation (and hence a CLT) is proved for $f \in L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ which satisfies

$$\sum_{k=1}^{\infty} \frac{\|E(S_k(f)\,|\,\mathcal{F}_0)\|_2}{k^{3/2}} < \infty, \qquad \sum_{k=1}^{\infty} \frac{\|S_k(f) - E(S_k(f)\,|\,\mathcal{F}_k)\|_2}{k^{3/2}} < \infty. \tag{3}$$

The idea of [10] is based on splitting of f into $f = f' + f''$ where $f' = E(f\,|\,\mathcal{F}_0)$ and applying an operator V which transforms the process $(U^i f'')$ into an adapted sequence $(U^i V f'')$. The assumption (3) then implies that both $(U^i f')$ and $(U^i V f'')$ satisfy (1). By the theorem of Maxwell and Woodroofe there exist martingale difference sequences $(U^i m')$ and $(U^i m'')$ adapted to $(\mathcal{F}_i)$ and approximating $(U^i f')$ and $(U^i f'')$ respectively. For $m = m' + m''$, $(U^i m)$ is then a martingale difference sequence for which (2) holds true.

As shown in [4], the operator V need not correspond to any point mapping and the method thus does not give directly an invariance principle.

In this paper we will present a generalisation of the Peligrad–Utev's maximal inequality to a larger class of processes, which will give a weak invariance principle for processes satisfying (3).

Let H be a subspace of L^2 for which $UH \subset H$. To the operator U we associate a semigroup of contraction operators P_{T^k}, $k = 1, 2, \ldots$, (recall that $Uf = f \circ T$) on H which satisfies:

(i) $P_{T^k} = P_T^k$, $k = 1, 2, \ldots$;
(ii) $P_T U = I$ where I is the identity operator;
(iii) if $P_T f = 0$ then $(U^i f)$ is a martingale difference sequence;

we denote $P_{T^1} = P_T = P$.

Proposition 1. *Let $f \in H$ be such that*

$$\sum_{k=1}^{\infty} \frac{\|\sum_{i=1}^{k} P^i f\|_2}{k^{3/2}} < \infty. \tag{4}$$

Then there exists a constant C such that for all $n \geqslant 1$,

$$\left\| \max_{1 \leqslant k \leqslant n} \left| \sum_{j=0}^{k-1} U^j f \right| \right\|_2 \leqslant C\sqrt{n}\left(\|f\|_2 + \sum_{k=1}^{n} \frac{\|\sum_{i=1}^{k} P^i f\|_2}{k^{3/2}} \right). \tag{5}$$

The proof is the same as the proof of Theorem 1 in [7]; in their case it can be taken $H = L^2(\mathcal{F}_0)$, $P_T f = E(Uf\,|\,\mathcal{F}_0)$, and U then replaced by U^{-1}. The inequality holds also in L^p spaces with $1 \leqslant p < \infty$ (cf. [7]). In [8], Proposition 1, Tyran-Kamińska and Mackey presented the proof in an operator language and proved the inequality for P_T being the Perron–Frobenius operator. This way the inequality was proved for noninvertible endomorphisms (e.g. exact endomorphisms, where no nontrivial martingale difference sequence $(U^i m)$ can exist). In the paper of Tyran-Kamińska and Mackey, T is a noninvertible endomorphism and the filtration is decreasing, given by $\mathcal{G}_i = T^{-i}\mathcal{A}$, $i \geqslant 0$. The endomorphism can, however be seen as a factor of an automorphism (cf. [2]); there thus exists a dynamical system $(\Omega_1, \mathcal{A}_1, \mu_1, T_1)$ where T_1 is an automorphism, a filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ with $\mathcal{F}_i \subset T^{-1}\mathcal{F}_i = \mathcal{F}_{i+1}$, such that $(\Omega, \mathcal{A}, \mu, T)$ is isomorphic to $(\Omega_1, \mathcal{F}_0, \mu_1, T_1^{-1})$. We take $H = L^2(\mathcal{F}_0)$ and define P_T by $P_T f = UE(f\,|\,\mathcal{F}_{-1}) = E(Uf\,|\,\mathcal{F}_0)$. The proposition above thus includes the case of Proposition 1 in [8].

Theorem 1. *Let $f \in L^2$ be regular, i.e. $\mathcal{F}_\infty$-measurable, $E(f\,|\,\mathcal{F}_{-\infty}) = 0$. If*

$$\sum_{k=1}^{\infty} \frac{\|S_k(f)\,|\,\mathcal{F}_0)\|}{k^{3/2}} < \infty, \qquad \sum_{k=1}^{\infty} \frac{\|S_k(f) - E(S_k(f)\,|\,\mathcal{F}_k)\|}{k^{3/2}} < \infty, \tag{6}$$

then the process of $w_n(t) = (1/\sqrt{n})\sum_{j=0}^{[nt]} U^j f$ weakly converges to the process $\eta^2 W$ where W is the Brownian motion and η^2 is independent of W.

Remark that if the measure μ is ergodic (i.e. for each A measurable, $A = T^{-1}A$ implies that A is either of measure 0 or of measure 1), η^2 is constant. In the nonergodic case we get η^2 constant on each ergodic component of μ (cf. [9]). In [8], a calculation of η^2 is given. For simplifying the notation we shall suppose that μ is ergodic.

For proving Theorem 1 we need to prove the central limit theorem for finite-dimensional distributions and the tightness (cf. [1]).

The central limit theorem for finite-dimensional distributions follows from (2) which has been proved in [10].

Let us define $f' = E(f \mid \mathcal{F}_0)$, $f'' = f - f'$. By the invariance principle of Peligrad and Utev (cf. [6]) we have the invariance principle for f'. It thus remains to prove the tightness for f''. It follows from the next proposition:

Proposition 2. *Let $f \in L^2$ be $\mathcal{F}_\infty$-measurable, $E(f \mid \mathcal{F}_0) = 0$, and*

$$\sum_{k=1}^{\infty} \frac{\|S_k(f) - E(S_k(f) \mid \mathcal{F}_k)\|}{k^{3/2}} < \infty. \tag{7}$$

Then the process of $w_n(t) = (1/\sqrt{n}) \sum_{j=0}^{[nt]} U^j f$ weakly converges to a Brownian motion.

Proof. Let $\mathcal{F}_i$ be a filtration with $\mathcal{F}_i \subset \mathcal{F}_{i+1} = T^{-1}\mathcal{F}_i$, P_{T^k}, $k = 1, 2, \ldots$, a set of operators on $H = L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_0)$ defined by

$$P_{T^k} h = U^{-k} h - E\big(U^{-k} h \mid \mathcal{F}_0\big).$$

We have $UH \subset H$ and we will prove that (i)–(iii) are fulfilled. Remark that

$$U^k E(f \mid \mathcal{F}_j) = E\big(U^k f \mid \mathcal{F}_{j+k}\big). \tag{8}$$

(i) For $k = 1$ the statement is true by definition, suppose that it is true for k.

$$\begin{aligned} P_T^{k+1} h &= P_T\big(U^{-k} h - E\big(U^{-k} h \mid \mathcal{F}_0\big)\big) \\ &= U^{-1}\big(U^{-k} h - E\big(U^{-k} h \mid \mathcal{F}_0\big)\big) - E\big(U^{-1}\big(U^{-k} h - E\big(U^{-k} h \mid \mathcal{F}_0\big)\big) \mid \mathcal{F}_0\big) \\ &= U^{-(k+1)} h - E\big(U^{-(k+1)} h \mid \mathcal{F}_0\big) = P_{T^{k+1}} h. \end{aligned}$$

(ii) From $h \in H$ it follows $E(h \mid \mathcal{F}_0) = 0$ hence $P_T U h = h - E(h \mid \mathcal{F}_0) = h$.

(iii) We get $0 = U P_T h$ hence by (8), $h = E(h \mid \mathcal{F}_1)$, i.e. h is $\mathcal{F}_1$-measurable. We have $h \in H$, hence $E(h \mid \mathcal{F}_0) = 0$, therefore $h \in L^2(\mathcal{F}_1) \ominus L^2(\mathcal{F}_0)$. Using (8) we get that $U^k h \in L^2(\mathcal{F}_{k+1}) \ominus L^2(\mathcal{F}_k)$ hence $(U^k h)$ is a martingale difference sequence.

From the fact that $\|S_k(f) - E(S_k(f) \mid \mathcal{F}_k)\|_2 = \|U^{-k}(S_k(f) - E(S_k(f) \mid \mathcal{F}_k))\|_2 = \|\sum_{j=1}^{k} P_T^j f\|_2$ we by Proposition 1 deduce the maximal inequality (5).

By [10] there is a martingale approximation (2) by a stationary martingale difference sequence and in the same way as in [6] or [8] we deduce the invariance principle. □

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