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Mathematical Problems in Mechanics

Corner instabilities in a slender nonlinearly elastic cylinder: analytical solutions and formation mechanism

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Abstract

In this Note, we study the corner instabilities in a slender cylinder constituted by a nonlinearly elastic material. Starting from the three-dimensional nonlinear field equations, we derive, through a novel method, a singular dynamical system as the normal form equation. It is shown that this system can capture the corner instabilities. We are also able to obtain analytical expressions of the solutions. The mechanism that causes corner formations is also found. ***To cite this article: H.-H. Dai, F.-F. Wang, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Instabilités en coin dans un cylindre mince non linéairement élastique : solutions analytiques mécanisme déformation. Dans cette Note, on étudie les instabilités « en coin » dans un cylindre mince formé d'un matériau non linéairement élastique. Partant des équations nonlinéaires tri-dimensionnelle, nous obtenons par une méthode nouvelle un système dynamique singulier. On montre que ce système retient les instabilités en coin. Nous obtenons également les expressions analytiques des solutions. On met aussi en évidence le fait que l'effet de couplage entre la nonlinéarité du matériau et la longueur caractéristique est le mécanisme qui provoque l'apparition de coins. ***Pour citer cet article : H.-H. Dai, F.-F. Wang, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

Stabilities and instabilities are important topics in nonlinearly elastic structures; see Antman [1]. There are few works on the bifurcations of the nonlinear field equations. One exception is the work by Healey and Montes-Pizarro [9], who utilized the generalized degree designed by Healey and Simpson [10] to give rigorous global bifurcation results for the compression of a three-dimensional nonlinearly elastic cylinder. In this note, we shall study the corner instabilities in a slender elastic cylinder based on the three-dimensional field equations. Here, a 'corner'

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means that the first-order derivative is discontinuous at one point. We shall use a novel method involving compound series-asymptotic expansions to carry out the analysis.

The present work is partially motivated by the Willis instability phenomenon occurring in a thick-walled elastic tube; see Beatty [2]. In that phenomenon, there is a corner formation in the interior surface of the tube. Actually, corner formations are widespread. Here, we intend to find the physical mechanism that causes the corner formations and provide analytical solutions. It will be shown that, mathematically, an appropriate singular dynamical system can be used to capture the corner solutions.

2. Normal formal equation of the nonlinear field equations

Let a and l denote the radius of the circular cylinder and its length. We use a cylindrical polar coordinate system and we denote by (R, Θ, Z) and (r, θ, z) the coordinates of a material point in the reference and current configurations respectively. The radial and axial displacements are denoted by $U(R, Z)$ and $W(R, Z)$. We consider a compressible isotropic hyperelastic Murnaghan material whose strain energy function Φ can be found in, e.g., [3]. The nominal stress tensor $\mathbf{S} = \frac{\partial \Phi}{\partial \mathbf{F}}$ (where $\mathbf{F}$ denotes the deformation gradient) satisfies the following field equations:

$$(S_{rR})_R + (S_{rZ})_Z + \frac{S_{rR} - S_{\theta\Theta}}{R} = 0, \qquad (S_{zR})_R + (S_{zZ})_Z + \frac{S_{zR}}{R} = 0. \tag{1}$$

We assume that the lateral surface of the cylinder is traction-free, i.e., $S_{rR}|_{R=a} = 0$ and $S_{zR}|_{R=a} = 0$.

Mathematically, to study the corner instabilities consists in studying the bifurcations of the complicated coupled nonlinear partial differential equations (1) and then showing that there are bifurcations leading to non-smooth solutions under the traction-free surface conditions with proper end conditions. However there is no available method for these tasks. Here, a novel approach involving compound series and asymptotic expansions is used to derive their normal form equation (cf. Dai and Huo [8], Dai and Fan [6] and Dai and Cai [5]). The computations are exceedingly delicate and are carried out by MATHEMATICA. Here we only give the main steps.

First we introduce a new set of dimensionless quantities through the following suitable scalings:

$$W = hw, \quad U = hu, \quad R = l\bar{r}, \quad Z = lx, \quad \varepsilon = \frac{h}{l}, \quad \delta = \frac{a^2}{l^2}, \tag{2}$$

where h is a characteristic axial displacement, and ε and δ are considered as small parameters. Further more we introduce a very important change of variables: $w = w(x, s)$, $u = \bar{r}v(x, s)$, $s = \bar{r}^2$ and $x = x$. The next step is to take series expansions of w and v in the neighborhood of $s = 0$:

$$w(s, x; \varepsilon, \delta) = w_0 + sw_1 + s^2 w_2 + \cdots, \qquad v(s, x; \varepsilon, \delta) = v_0 + sv_1 + s^2 v_2 + \cdots. \tag{3}$$

Substituting (3) into the field equations, we can obtain two equations which contain terms with $s^0, s^1, \ldots$. Letting the coefficients of s^0, s^1,... vanish and combing with the boundary conditions, we obtain five equations with five unknowns w_0, w_1, w_2, v_0 and v_1. By using asymptotic expansions in ε, the unknowns v_0, v_1, w_1 and w_2 can be expressed in terms of w_{0x}. In this fashion we obtain a governing equation for εw_{0x}:

$$V + D_1 V^2 - a^2 \left(\frac{1}{4} V_{ZZ} + D_2 V_Z^2 + 2D_2 V V_{ZZ} \right) = \gamma, \tag{4}$$

where $V = \varepsilon w_{0x}$ and D_1 and D_2 are constants related to constitutive constants and γ is the dimensionless averaged end stress. Eq. (4) is called *the normal form equation* of the system with a given γ, since it contains all the required terms to yield the leading-order behavior of the original system.

3. Solutions

We can rewrite (4) as a first-order system, viz.,

$$V_Z = y, \quad y_Z = \frac{V + D_1 V^2 - a^2 D_2 y^2 - \gamma}{a^2(\frac{1}{4} + 2D_2 V)}. \tag{5}$$

This is a singular dynamical system since there is a vertical singular line $V = -\frac{1}{8D_2}$ in the phase plane.

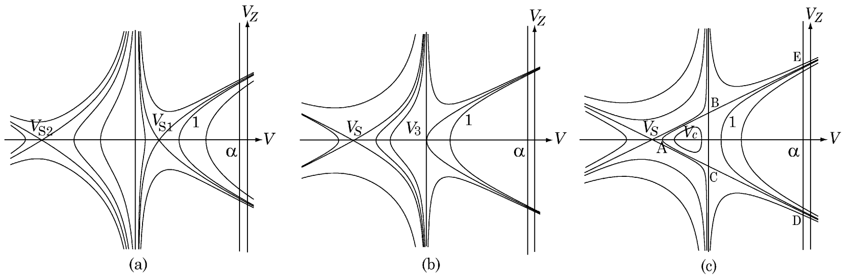

Fig. 1. Phase planes for different γ values: (a) $\gamma_p < \gamma < 0$; (b) $\gamma = \gamma_p$; (c) $\gamma_c \leqslant \gamma < \gamma_p$.

In the following, we study two boundary-value problems of this system under 'sliding'(or natural) boundary conditions and clamped boundary conditions. Although governing traveling waves for such systems have been studied in the past (see Dai [4] and Dai and Huo [7]), it appears that *the boundary-value problem of a singular dynamical system was never formulated nor studied before*. Therefore *we have a new mathematical problem at hand*.

Here we only consider the case $4D_2 > D_1 > 0$ since the other cases can be similarly considered. As γ increases, there are several different phase planes, which are shown in Fig. 1.

Case I. Natural Boundary Conditions

In this case, $V_Z = 0$ at $Z = 0$ and l. Without loss of generality, we let $l = 1$. In the phase plane for case(a) or (b), there are two trivial solutions, and the trivial solution with a smaller strain value has the smaller energy value, which is the stable configuration. In the phase plane of case (c), in addition to two trivial solutions, there exist nontrivial solutions, for which we have the following theorem:

Theorem 3.1. *In the phase plane* (c)*, there are two types of nontrivial solutions* (*one being periodic and one being non-smooth*)*, for which the following results hold.*

(1) *There exists a periodic solution if and only if there exists a positive integer n such that*

$$n = \beta \frac{\sqrt{(-\frac{1}{8D_2} - E_2)(E_3 - E_1)}}{2(-\frac{1}{8D_2} - E_3)} \frac{1}{\Pi(\frac{\pi}{2}, \frac{E_3 - E_2}{-1/8D_2 - E_2}, q)}, \tag{6}$$

where

$$\beta = \frac{1}{a}\sqrt{\frac{D_1}{3D_2}}, \qquad q = \sqrt{\frac{(E_2 - E_3)(E_1 + \frac{1}{8D_2})}{(E_1 - E_3)(E_2 + \frac{1}{8D_2})}}, \tag{7}$$

where Π is the elliptic integral of the third kind and E_i $(i = 2, 3)$ are the points where the trajectory crosses the V-axis, and E_1 can be expressed in terms of E_2, D_1, D_2 and γ.

(2) *The non-smooth* (*corner*) *solution occurs at the critical stress value*

$$\gamma_c = \frac{(D_1^2 - 8D_1D_2 - 48D_2^2)(1 + \mathrm{e}^{2\beta}) + (14D_1^2 - 112D_1D_2 + 96D_2^2)\mathrm{e}^{\beta}}{256D_1D_2^2(1 + \mathrm{e}^{\beta})^2}, \tag{8}$$

and its explicit solution is given by

$$V = E_2 + (E_2 - E_1)\sinh^2\left(\operatorname{arcsinh}\sqrt{\frac{-\frac{1}{8D_2} - E_2}{-E_1 + E_2}} - \frac{1}{2}\beta\left|Z - \frac{1}{2}\right|\right), \quad 0 \leqslant Z \leqslant 1, \tag{9}$$

where E_1 and E_2 can be expressed in terms of γ, D_1, D_2 and a.

In the phase plane (c), when $\gamma = \gamma_c$, there is also a corner solution in addition to two trivial solutions and periodic solutions. It starts from A, goes to B, then jumps from B to C and finally goes back to A. If D_1, D_2 and the Poisson ratio ν are properly chosen, this corner solution has the smallest total potential energy value and thus it represents a stable configuration.

Case II. Clamped Boundary Conditions

In this case the radial displacements at the points $(0, a)$ and $(1, a)$ vanish. After some calculations, it is found that this condition is equivalent to $V = \alpha$ at $Z = 0$ and 1, where α is a negative constant.

For the cases $\gamma_p < \gamma < 0$, $\gamma = \gamma_p$ and $\gamma_c < \gamma < \gamma_p$, there is a unique solution indexed by 1 in the corresponding phase plane. At a critical stress value $\gamma = \gamma_c$, in addition to the solution 1, there is also a corner solution. It starts from D, goes to C, then jumps from C to B and finally arrives at E. We then have the following theorem:

Theorem 3.2. *Trajectory* 1 *is a solution if and only if there exists a value V_0 such that*

$$\frac{1}{2} = \frac{1}{\beta}\int_{V_0}^{\alpha}\sqrt{\frac{\tau + \frac{1}{8D_2}}{(\tau - V_0)(\tau^2 + s\tau + t)}}\,\mathrm{d}\tau, \quad \textit{where } s = \frac{3 + 2D_1V_0}{2D_1} \textit{ and } t = \frac{-6\gamma + 3V_0 + 2D_1V_0^2}{2D_1}. \tag{10}$$

There exists a corner solution in the phase plane (c) *if and only if there exists a critical value $\gamma = \gamma_c$ such that*

$$\frac{1}{2} = \frac{1}{\beta}\left(2\,\mathrm{arcsinh}\sqrt{\frac{\alpha - E_2}{-E_1 + E_2}} - 2\,\mathrm{arcsinh}\sqrt{\frac{-\frac{1}{8D_2} - E_2}{-E_1 + E_2}}\right), \tag{11}$$

where E_1 and E_2 are two constants expressed in terms of γ, D_1, D_2 and a. Its solution is given by

$$V = E_2 + (E_2 - E_1)\sinh^2\left(\mathrm{arcsinh}\sqrt{\frac{-\frac{1}{8D_2} - E_2}{-E_1 + E_2}} + \frac{1}{2}\beta\left|\frac{1}{2} - Z\right|\right), \quad 0 \leqslant Z \leqslant 1. \tag{12}$$

If D_1, D_2 and the Poisson ratio ν are properly chosen, this corner solution exists and has the smaller total potential energy value. It thus represents a stable configuration.

The difference between the two cases is that it is *an inward corner* in Case I and *an outward corner* in Case II. So, the end conditions have a great influence on the bifurcations.

In conclusion, we find that *mathematically a singular dynamical system can indeed capture the corner solutions*. The correspond singularity comes from the zero in the denominator of (5), which is due to the coupling effect of the material nonlinearity and the geometrical size. Thus, our results reveal that *the mechanism causing the corner instabilities is the interaction between the material nonlinearity and the geometrical size*.

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