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Differential Geometry/Dynamical Systems

Periodic orbits in the case of a zero eigenvalue

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Abstract

We will show that if a dynamical system has enough constants of motion then a Moser–Weinstein type theorem can be applied for proving the existence of periodic orbits in the case when the linearized system is degenerate. ***To cite this article: P. Birtea et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Sur l'existence d'orbites périodiques en cas de valeur propre zéro. On va montrer que si un système dynamique a assez d'intégrales premières, alors on peut utiliser un théorème de type Moser–Weinstein pour prouver l'existence d'orbites périodiques, même si le système linéarisé associé est dégénéré. ***Pour citer cet article : P. Birtea et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Pour étudier l'existence d'orbites périodiques d'un système d'équations différentielles ordinaires, en utilisant le théorème de Moser [3], le système linéarisé autour d'un point d'équilibre doit être nondégénéré. Dans le cas dégénéré, si on trouve assez d'intégrales premières, on obtient un résultat similaire à ce théorème, donné par :

Théorème 0.1. *Soit $\dot{x} = X(x)$ un système dynamique, x_0 un point d'équilibre, i.e., $X(x_0) = 0$ et $C := (C_1, \dots, C_k) : M \to \mathbb{R}^k$ une intégrale première vectorielle du système, avec $C(x_0)$ une valeur régulière de C. Si*

(i) *l'espace propre correspondant à la valeur propre zéro du système linéarisé autour de x_0 a la dimension k,*
(ii) *$DX(x_0)$ a une paire de valeurs propres imaginaires $\pm i\omega$, $\omega \neq 0$,*
(iii) *il existe une intégrale première $I : M \to \mathbb{R}$ de X, avec $\mathrm{d}I(x_0) = 0$ telle que*

$$\mathrm{d}^2 I(x_0)_{|W \times W} > 0,$$

où $W = \bigcap_{i=1}^{k} \ker \mathrm{d}C_i(x_0)$,

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alors, pour chaque $\varepsilon \in \mathbb{R}$ suffisamment petit, toutes les surfaces intégrales

$$I(x) = I(x_0) + \varepsilon^2$$

contiennent au moins une orbite périodique de X.

1. Introduction

Finding periodic solutions of a system of ordinary differential equations is an old problem in mathematical physics going back to Lyapunov and Poincaré. Periodic solutions were discovered first for linear conservative systems that appear in mechanics. The passage from linear to nonlinear systems was taken by Lyapunov [2] under the assumption of existence of an integral of motion and a certain nonresonance condition.

In 1973, Weinstein [4] proved that in the case of a Hamiltonian system with a positive definite Hamiltonian function the nonresonance condition is not necessary. Later, Moser [3] extended Weinstein's result to the case of a general dynamical system which possess a constant of motion. More precisely, let

$$\dot{x} = X(x), \tag{1}$$

be a dynamical system generated by the C^1 vector field X on a differentiable manifold M with x_0 an equilibrium point, i.e., $X(x_0) = 0$. Consider the linearized equations for the equilibrium point x_0,

$$\dot{z} = DX(x_0) \cdot z. \tag{2}$$

Then we have the following result due to Moser [3]:

Theorem *(Moser). Let $I \in C^2$ be an integral of motion for* (1) *with* $\mathrm{d}I(x_0) = 0$. *If*

(i) *$DX(x_0)$ is a nonsingular matrix,*
(ii) *$DX(x_0)$ has a pair of imaginary eigenvalues $\pm \mathrm{i}\omega$ with $\omega \neq 0$,*
(iii) $\mathrm{d}^2 I(x_0)$ *is positive definite,*

then for sufficiently small $\varepsilon \in \mathbb{R}$ any integral surface

$$I(x) = I(x_0) + \varepsilon^2$$

contains at least one periodic solution of X whose period is close to the period of the corresponding linear system around x_0.

The condition (i) of the above theorem implies that the linearized system around the critical point x_0 cannot have a zero eigenvalue. This restriction makes the theorem unapplicable to a series of examples. We will show that in the case when for (1) one can find enough constants of motion a similar result can be applied for proving the existence of periodic orbits. We will also illustrate this with two examples.

2. The main result

Theorem 2.1. *Let $\dot{x} = X(x)$ be a dynamical system, x_0 an equilibrium point, i.e., $X(x_0) = 0$ and $C := (C_1, \ldots, C_k) : M \to \mathbb{R}^k$ a vector valued constant of motion for the above dynamical system with $C(x_0)$ a regular value for C. If*

(i) *the eigenspace corresponding to the eigenvalue zero of the linearized system around x_0 has dimension k,*
(ii) *$DX(x_0)$ has a pair of imaginary eigenvalues $\pm \mathrm{i}\omega$ with $\omega \neq 0$,*
(iii) *there exist a constant of motion $I : M \to \mathbb{R}$ for the vector field X with* $\mathrm{d}I(x_0) = 0$ *and such that*

$$\mathrm{d}^2 I(x_0)_{|W \times W} > 0,$$

where $W = \bigcap_{i=1}^{k} \ker \mathrm{d}C_i(x_0)$, *then for each sufficiently small $\varepsilon \in \mathbb{R}$, any integral surface*

$$I(x) = I(x_0) + \varepsilon^2$$

contains at least one periodic solution of X whose period is close to the period of the corresponding linear system around x_0.

Proof. If $C_i \in C^\infty(M, \mathbb{R})$ is a constant of motion for the dynamic generated by the vector field X then $DX(x_0)\nabla C_i(x_0) = 0$, and hence $\nabla C_i(x_0) \in \ker DX(x_0)$.

Because $C(x_0)$ is a regular value for C we have that $\nabla C_i(x_0)$, $i = \overline{1,k}$ are linearly independent vectors in the tangent space $T_{x_0}M$. Then, hypothesis (i) and the fact that $C_1, \ldots, C_k \in C^\infty(M, \mathbb{R})$ are constants of motion for X implies the following equality,

$$\operatorname{span}_{\mathbb{R}}\left\{\nabla C_i(x_0)\colon\ i = \overline{1,k}\right\} = \ker DX(x_0)\ (= V_{\lambda=0}),$$

where $V_{\lambda=0}$ is the eigenspace corresponding to the zero eigenvalue of the matrix which is canonically associated to the linear part at the equilibrium of interest x_0 of our system determined by X.

This argument implies that the reduced system

$$\begin{cases} \dot{x} = X(x), \\ C(x) = C(x_0), \end{cases}$$

which is the original system restricted to the submanifold $C^{-1}(C(x_0))$ has the linearization about x_0 without eigenvalue zero.

The function $I_{|C^{-1}(C(x_0))} : C^{-1}(C(x_0)) \to \mathbb{R}$ is a first integral for the reduced system with $\mathrm{d}(I_{|C^{-1}(C(x_0))})(x_0) = 0$ and hypothesis (iii) obviously implies that $\mathrm{d}^2(I_{|C^{-1}(C(x_0))})(x_0) > 0$. By the Moser theorem we have that for sufficiently small $\varepsilon \in \mathbb{R}$, any integral surface

$$I(x) = I(x_0) + \varepsilon^2$$

contains at least one periodic solution of the reduced system and hence of the initial system. □

Remark 2.1. If the dynamic system (1) is Hamilton–Poisson and x_0 is regular point in the sense that it is contained in a maximal dimension symplectic leaf of $(M, \{\cdot\,,\cdot\})$ which is determined by the Casimirs $C_1, \ldots, C_k$, then by the theorem of Weinstein [4] one has the existence of at least $(\dim P - k)/2$ periodic orbits.

3. Examples

3.1. Rigid body with one control

Let us consider the rigid body dynamics with one control,

$$\begin{cases} \dot{m}_1 = a_1 m_2 m_3, \\ \dot{m}_2 = a_2 m_1 m_3, \\ \dot{m}_3 = (a_3 - l) m_1 m_2 \end{cases} \tag{3}$$

where $l \in \mathbb{R}$ is the gain parameter.

Let us make use the following notation $\alpha := (a_3 - l)/a_3$. Then it is not hard to see that our dynamics (3) has the following Hamilton–Poisson realization $(\mathbb{R}^3, \Pi_\alpha, H_\alpha)$, where

$$\Pi_\alpha \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -m_3 & \alpha m_2 \\ m_3 & 0 & -\alpha m_1 \\ -\alpha m_2 & \alpha m_1 & 0 \end{bmatrix}$$

is the Poisson structure and $H_\alpha(m_1, m_2, m_3) \stackrel{\text{def}}{=} \frac{1}{2}(m_1^2/I_1 + m_2^2/I_2 + m_3^2/\alpha I_3)$ is the Hamiltonian function. Moreover, the smooth function $C_\alpha \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by

$$C_\alpha(m_1, m_2, m_3) \stackrel{\text{def}}{=} \alpha m_1^2 + \alpha m_2^2 + m_3^2$$

is a Casimir of our Poisson configuration $(\mathbb{R}^3, \Pi_\alpha)$.

Let us concentrate now on the equilibrium state

$$e_1^M = (M, 0, 0), \quad M \in \mathbb{R}^*$$

of our dynamics (3). Then, under the restriction $l < a_3$, we have successively:

(i) The restriction of the dynamics (3) to the coadjoint orbit

$$\alpha m_1^2 + \alpha m_2^2 + m_3^2 = \alpha M^2 \tag{4}$$

gives rise to a Hamiltonian system on a symplectic manifold.

(ii) $\operatorname{span}_{\mathbb{R}} (\nabla C_\alpha(e_1^M)) = V_{\lambda=0} = \operatorname{span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ where

$$V_{\lambda=0} = \left\{ \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; A\big(e_1^M\big) \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

$A(e_1^M)$ being the matrix of the linear part of the dynamics (3) at the equilibrium of interest e_1^M, $M \in \mathbb{R}^*$.

(iii) The matrix of the linear part of our reduced dynamics to (4) has at the equilibrium e_1^M the following characteristic roots:

$$\lambda_{1,2} = \pm M\mathrm{i}\sqrt{-a_2(a_3 - l)}.$$

(iv) The smooth function $F_{\frac{1}{\alpha I_1}} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by:

$$F_{\frac{1}{\alpha I_1}}(m_1, m_2, m_3) = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{\alpha I_3}\right) - \frac{1}{2\alpha I_1}\big(\alpha m_1^2 + \alpha m_2^2 + m_3^2\big)$$

is a constant of motion and e_1^M is a local minimum of $F_{\frac{1}{\alpha I_1}}$ with the constraint (4).

Then via Theorem 2.1 we have:

Proposition 3.1. *If $l < a_3$ then the reduced dynamics to the coadjoint orbit* (4) *has near the equilibrium state e_1^M, $M \in \mathbb{R}^*$ at least one periodic solution whose period is close to*

$$\frac{2\pi}{|M|\sqrt{-a_2(a_3 - l)}}.$$

Remark 3.1. Similar results can be also obtained for the equilibrium states

$$e_2^M = (0, M, 0), \quad M \in \mathbb{R}^*$$

and

$$e_3^M = (0, 0, M), \quad M \in \mathbb{R}^*.$$

3.2. Clebsch system

It is well known that the Clebsch system can be written in the following form:

$$\begin{cases} \dot{x}_1 = x_2 p_3 - x_3 p_2, \\ \dot{x}_2 = x_3 p_1 - x_1 p_3, \\ \dot{x}_3 = x_1 p_2 - x_2 p_1, \\ \dot{p}_1 = (a_3 - a_2) x_2 x_3, \\ \dot{p}_2 = (a_1 - a_3) x_1 x_3, \\ \dot{p}_3 = (a_2 - a_1) x_1 x_2, \end{cases} \tag{5}$$

where

$$a_1, a_2, a_3 \in \mathbb{R},$$
$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0,$$
$$a_1 \neq a_2 \neq a_3$$

(see for details Dubrovin, Krichever and Novikov [1]).

It is not hard to see that the smooth functions $H, C, D \in C^\infty(\mathbb{R}^6, \mathbb{R})$ given by:

$$\begin{aligned}
&H(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}\big(a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + p_1^2 + p_2^2 + p_3^2\big),\\
&C(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}\big(x_1^2 + x_2^2 + x_3^2\big),\\
&D(x_1, x_2, x_3, p_1, p_2, p_3) = x_1p_1 + x_2p_2 + x_3p_3
\end{aligned}$$

are constants of motion for the Clebsch system.

Let us concentrate now to the equilibrium state $e_1^M = (M, 0, 0, 0, 0, 0)$, $M \in \mathbb{R}^*$. Then under the restrictions:

$$a_1 < \min\{a_2, a_3\},$$

we have successively,

(i) $\mathrm{span}_{\mathbb{R}}(\nabla C(e_1^M), \nabla D(e_1^M)) = V_{\lambda=0}$ where

$$V_{\lambda=0} = \left\{ \begin{bmatrix} m_1\\ m_2\\ m_3\\ p_1\\ p_2\\ p_3 \end{bmatrix} \in \mathbb{R}^6 \;\middle|\; A\big(e_1^M\big) \begin{bmatrix} m_1\\ m_2\\ m_3\\ p_1\\ p_2\\ p_3 \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0\\0\\0 \end{bmatrix} \right\},$$

$A(e_1^M)$ being the matrix of the linear part of the dynamics (5) at the equilibrium e_1^M.

(ii) The matrix of the linear part of our reduced dynamics to the constraint

$$\left\{ (x_1, x_2, x_3, p_1, p_2, p_3) \in \mathbb{R}^6 \;\middle|\; \begin{array}{l} x_1^2 + x_2^2 + x_3^2 = M^2 \\ x_1p_1 + x_2p_2 + x_3p_3 = 0 \end{array} \right\}, \tag{6}$$

at the equilibrium e_1^M has the following characteristic roots:

$$\begin{aligned}
&\lambda_{1,2} = \pm \mathrm{i} M\sqrt{a_3 - a_1},\\
&\lambda_{3,4} = \pm \mathrm{i} M\sqrt{a_2 - a_1}.
\end{aligned}$$

(iii) The smooth function $F_{a_1} \in C^\infty(\mathbb{R}^6, \mathbb{R})$ given by:

$$F_{a_1}(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2}\big(a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + p_1^2 + p_2^2 + p_3^2\big) - \frac{a_1}{2}\big(x_1^2 + x_2^2 + x_3^2\big)$$

is a constant of motion and e_1^M is a local minimum of F_{a_1} with the constraint (6).

Then via Theorem 2.1 we have:

Proposition 3.2. *If* $a_1 \in\,]a_2, a_3[$ *then the reduced dynamics to* (6) *has near* e_1^M*,* $M \in \mathbb{R}^*$ *at least one periodic solution.*

Remark 3.2. Similar results can be also obtained for the equilibrium states:

$$e_2^M = (0, M, 0, 0, 0, 0), \quad M \in \mathbb{R}^*,$$

and

$$e_3^M = (0, 0, M, 0, 0, 0), \quad M \in \mathbb{R}^*.$$

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