Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 344 (2007) 721–726
COMPTES RENDUS MATHEMATIQUE
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Mathematical Problems in Mechanics

Weakened conditions of admissibility of surface forces applied to linearly elastic membrane shells

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Received 17 March 2007; accepted 19 April 2007

Available online 1 June 2007

Presented by Philippe G. Ciarlet

Abstract

In this Note, we consider linearly elastic generalized membrane shells. In order to obtain the convergence of the scaled displacements as the thickness 2ε approaches zero, we define new conditions of admissibility when the forces are applied on the surface of the shell. They are independent of ε, which allows us to obtain sufficient conditions rather easily. ***To cite this article: R. Luce et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Conditions affaiblies d'admissibilité pour des densités surfaciques de forces appliquées à des coques en membrane linéairement élastiques. Dans cette Note, on se place dans le cadre de l'élasticité linéaire pour les coques en membrane généralisées. Afin d'obtenir la convergence des déplacements normalisés quand l'épaisseur 2ε tend vers zéro, nous définissons de nouvelles conditions d'admissibilité pour des forces surfaciques. Ces conditions sont indépendantes de ε, ce qui permet d'obtenir relativement facilement des conditions suffisantes d'admissibilité. ***Pour citer cet article : R. Luce et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Dans un domaine ω de $\mathbb{R}^2$, considérons une carte $\theta : \overline{\omega} \to \mathbb{R}^3 \in C^3(\overline{\omega}; \mathbb{R}^3)$, injective, telle que, pour $\alpha = 1, 2$, les deux vecteurs $a_\alpha = \partial_\alpha \theta(y)$ soient linéairement indépendants en tout point $y \in \overline{\omega}$. On lui associe la famille de coques en membrane généralisées linéairement élastiques de type 1 c'est à dire en flexion pure inhibée (voir définition dans Ciarlet [2] p. 262 ou dans Sanchez-Hubert et Sanchez-Palencia [6]) de surface moyenne $S = \theta(\overline{\omega})$, d'épaisseur 2ε. On la soumet à des densités de forces extérieures $\mathbf{f}$, sachant que les conditions aux limites de type Dirichlet homogène sont posées sur une même portion du bord latéral $\theta(\gamma_0)$. En faisant tendre l'épaisseur vers zéro, on se ramène à un problème variationnel posé sur S, bien posé lorsque $\mathbf{f}$ appartient au dual de l'espace où on cherche les déplacements (voir Chapelle et Bathe [1], p. 129 ou Ciarlet [2]).

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.04.020

Ph. Ciarlet et V. Lods ont démontré de plus que, si les forces surfaciques et volumiques sont « admissibles », la solution $\mathbf{u}(\varepsilon)$ de (1), problème variationnel normalisé 3D associé, converge fortement dans un espace obtenu par complétion, et que sa valeur moyenne dans l'épaisseur, $\overline{\mathbf{u}(\varepsilon)}$, converge fortement également, dans un espace complété, vers l'unique solution ζ du problème variationnel (2) posé cette fois-ci sur une surface (voir [3]). Mais, comme Ph. Ciarlet l'a fait remarquer dans [2] p. 293, "*... l'identification des forces admissibles demande généralement une analyse délicate ...*".

Dans cette Note, nous nous sommes attachés à simplifier les conditions d'admissibilité dans le cas où les forces extérieures sont exclusivement surfaciques. Le Théorème 2 montre que les conditions d'admissibilité, données dans [2], p. 265 et rappelées dans le Théorème 1 pour des forces uniquement surfaciques, peuvent être considérablement affaiblies. En effet nous donnons maintenant dans (3) des conditions d'admissibilité indépendantes de ε. Ce qui permet par la suite d'identifier plus facilement des conditions suffisantes d'admissiblité, en donnant par exemple des conditions d'existence de solutions d'un système d'EDP comme dans le cas particulier traité dans Poutous [5].

Signalons d'autre part, que dans le cas de forces non admissibles, V. Lods et C. Mardare ont montré, sous réserve que la coque soit totalement encastrée, la convergence, dans une norme liée à l'énergie, de $\mathbf{u}(\varepsilon)$ vers le déplacement obtenu par les modèles linéaires de Koiter et de Naghdi (voir [4]).

1. Introduction and notations

In this Note, Greek indices take their values in $\{1,2\}$, whereas Latin indices belong to $\{1,2,3\}$ and the repeated index summation convention is used. Let us first consider the '2D' ill-posed scaled variational problem

$$\mathcal{P}(\omega):\quad \begin{cases} \zeta \in \mathbf{V}(\omega) := \{\eta = (\eta_i) \in \mathbf{H}^1(\omega);\ \eta = \mathbf{0} \text{ on } \gamma_0\}, & \forall \eta \in \mathbf{V}(\omega) \\ \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\zeta)\gamma_{\alpha\beta}(\eta)\sqrt{a}\,\mathrm{d}y = \int_\omega h^i v_i \sqrt{a}\,\mathrm{d}y \end{cases}$$

where the bilinear form is not coercive on $\mathbf{V}(\omega)$, the surface functions $h^i \in L^2(\omega)$ are independent of ε, ω is a domain in $\mathbb{R}^2$ (open, bounded, connected subset with a Lipschitz-continuous boundary, the set ω being locally on one side of its boundary), $\theta : \overline{\omega} \to \mathbb{R}^3 \in C^3(\overline{\omega}; \mathbb{R}^3)$ is an injective mapping such that the two vectors $\mathbf{a}_\alpha := \partial_\alpha \theta(y)$ are linearly independent at each point $y \in \overline{\omega}$, where $\mathbf{a}_3 := \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|}$, and $a_{\alpha\beta} := \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ denote the covariant components of the metric tensor of $S := \theta(\overline{\omega})$, and $a := \det((a_{\alpha\beta})_{\alpha\beta})$, $a^{\alpha\beta}$ denote the contravariant components of the metric tensor of $S := \theta(\overline{\omega})$, where $a^{\alpha\beta\sigma\tau}$ which denote the contravariant components of the scaled 2D elasticity tensor are defined by

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu\big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\big) \quad \text{with } \lambda > 0,\ \mu > 0$$

where $\Gamma^\sigma_{\alpha\beta}$ are the surface Christoffel symbols i.e. $\Gamma^\sigma_{\alpha\beta} := \mathbf{a}^\sigma \cdot \partial_\alpha \mathbf{a}_\beta$ with $\mathbf{a}^i \cdot \mathbf{a}_j = \delta_{ij}$, and where, for any vector field $\eta = (\eta_i) \in \mathbf{H}^1(\omega)$, the covariant components of the 2D linearized change of metric tensor $\gamma_{\alpha\beta}(\eta) \in L^2(\omega)$ are defined by

$$\gamma_{\alpha\beta}(\eta) := \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3 \quad \text{with } b_{\alpha\beta} := \mathbf{a}_3 \cdot \partial_\alpha \mathbf{a}_\beta.$$

Let us also consider the 3D scaled variational problem

$$\mathcal{P}(\varepsilon; \Omega):\quad \begin{cases} \mathbf{u}(\varepsilon) \in \mathbf{V}(\Omega) := \{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega);\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0 := \gamma_0 \times [-1,1]\}, & \forall \mathbf{v} \in \mathbf{V}(\Omega) \\ \int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon; \mathbf{u}(\varepsilon)) e_{i\|j}(\varepsilon; \mathbf{v})\sqrt{g(\varepsilon)}\,\mathrm{d}x = \int_{\Gamma^+\cup\Gamma^-} h^{i\pm} v_i \sqrt{g(\varepsilon)}\,\mathrm{d}\Gamma \end{cases} \tag{1}$$

where the functions $h^{i\pm} \in L^2(\Gamma^+ \cup \Gamma^-)$ are independent of ε, $\Omega := \omega\times]-1,1[$, $\Gamma^+ := \omega \times \{1\}$, $\Gamma^- := \omega \times \{-1\}$, and $\Omega_\varepsilon := \omega\times]-\varepsilon,\varepsilon[$, $\boldsymbol{\Theta} : \overline{\Omega}_\varepsilon \to \mathbb{R}^3$ is the canonical extension of θ and thus verifies $\boldsymbol{\Theta}(y, x_3) := \theta(y) + x_3 a_3$ and $\det(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) > 0$ (where $\mathbf{g}_i := \partial_i \boldsymbol{\Theta}$), where, for any vector field $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$, the scaled linearized strains $e_{i\|j}(\varepsilon; \mathbf{v}) = e_{j\|i}(\varepsilon; \mathbf{v}) \in \mathbf{L}^2(\Omega)$ are defined by

$$e_{\alpha\|\beta}(\varepsilon; \mathbf{v}) = \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma^p_{\alpha\beta}(\varepsilon) v_p, \qquad e_{\alpha\|3}(\varepsilon; \mathbf{v}) := \frac{1}{2}\left(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3\right) - \Gamma^\sigma_{\alpha3}(\varepsilon) v_\sigma,$$

$$e_{3\|3}(\varepsilon; \mathbf{v}) := \frac{1}{\varepsilon}\partial_3 v_3$$

with $\Gamma_{ij}^{p}(\varepsilon):\overline{\Omega}\to\mathbb{R}$ being the scaled 3D Christoffel symbols i.e.

$$\Gamma_{ij}^{p}(\varepsilon)(x_1,x_2,x_3):=\Gamma_{ij}^{\varepsilon,p}(x_1,x_2,\varepsilon x_3)\quad\text{and}\quad \Gamma_{ij}^{\varepsilon,p}:=\mathbf{g}^p\cdot\partial_i\mathbf{g}_j\quad\text{with }\mathbf{g}^i\cdot\mathbf{g}_j=\delta_{ij},$$

with also, $g(\varepsilon):\overline{\Omega}\to\mathbb{R}$ being the scaled function of $g^\varepsilon:=\det(\mathbf{g}_i\cdot\mathbf{g}_j)$, i.e., $g(\varepsilon)(x_1,x_2,x_3):=g^\varepsilon(x_1,x_2,\varepsilon x_3)$, and where, at last, the contravariant components $A^{ijkl}(\varepsilon):\overline{\Omega}\to\mathbb{R}$ of the scaled 3D elasticity tensor satisfy

$$A^{ijkl}(\varepsilon)=A^{jikl}(\varepsilon)=A^{klij}(\varepsilon),\quad A^{ijkl}(\varepsilon)=A^{ijkl}(0)+\mathrm{O}(\varepsilon)\quad\text{and}\quad A^{\alpha\beta\sigma 3}(\varepsilon)=A^{\alpha 333}(\varepsilon)=0,$$

where the order symbol is meant with respect to the norm $\|w\|_{0,\infty,\overline{\Omega}}:=\sup\{|w(x)|,x\in\overline{\Omega}\}$ and

$$A^{\alpha\beta\sigma\tau}(0):=\lambda a^{\alpha\beta}a^{\sigma\tau}+\mu\big(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}\big),\quad A^{\alpha\beta 33}(0):=\lambda a^{\alpha\beta},\quad A^{\alpha 3\sigma 3}(0):=\mu a^{\alpha\sigma},$$
$$A^{3333}(0):=\lambda+2\mu,\quad A^{\alpha\beta\sigma 3}(0)=A^{\alpha 333}(0):=0.$$

Let us now assume that the semi norm $|\cdot|_\omega^M$ defined by $|\eta|_\omega^M:=\{\sum_{\alpha,\beta}|\gamma_{\alpha\beta}(\eta)|_{0,\omega}^2\}^{1/2}$ is a norm over the space $\mathbf{V}(\omega)$ which is not equivalent to the norm $\|\cdot\|_{1,\omega}$ and let $\mathbf{V}_M^\#(\omega)$ be the completion of $\mathbf{V}(\omega)$ with respect to $|\cdot|_\omega^M$. Let $|\cdot|_\Omega^M$ be the norm over $\mathbf{V}(\Omega)$ defined by

$$|\mathbf{v}|_\Omega^M=\big\{|\partial_3\mathbf{v}|_{0,\omega}^2+\big(|\bar{\mathbf{v}}|_\omega^M\big)^2\big\}^{1/2}\quad\text{where }\bar{\mathbf{v}}:=\frac{1}{2}\int_{-1}^{1}\mathbf{v}\,\mathrm{d}x_3$$

and let $\mathbf{V}_M^\#(\Omega)$ be the completion of $\mathbf{V}(\Omega)$ with respect to $|\cdot|_\Omega^M$.

Let $B_M(\zeta,\eta):=\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\zeta)\gamma_{\alpha\beta}(\eta)\sqrt{a}\,\mathrm{d}y$ and $L_M(\eta):=\int_\omega h^i\eta_i\sqrt{a}\,\mathrm{d}y$ and let $B_M^\#$ and $L_M^\#$ denote the unique continuous extensions from $\mathbf{V}(\omega)$ to $\mathbf{V}_M^\#(\omega)$ of the bilinear form B_M and the linear form L_M.

Under all these assumptions, Ph. Ciarlet proved that

Theorem 1. *There exist* $\mathbf{u}$ *in* $\mathbf{V}_M^\#(\Omega)$ *and* ζ *in* $\mathbf{V}_M^\#(\omega)$ *such that*

$$\mathbf{u}(\varepsilon)\longrightarrow\mathbf{u}\quad\text{in }\mathbf{V}_M^\#(\Omega)\text{ as }\varepsilon\to 0\quad\text{and}\quad\overline{\mathbf{u}(\varepsilon)}\longrightarrow\zeta\quad\text{in }\mathbf{V}_M^\#(\omega)\text{ as }\varepsilon\to 0$$

and the limit ζ *satisfies the scaled* 2D *variational problem of a linearly elastic generalized membrane shell of the first kind*

$$\mathcal{P}_{\mathcal{M}}^\#(\omega):\quad\begin{cases}\zeta\in\mathbf{V}_M^\#(\omega),\quad\forall\eta\in\mathbf{V}_M^\#(\omega)\\ B_M^\#(\zeta,\eta)=L_M^\#(\eta)\end{cases}\tag{2}$$

if the density of surface force h is admissible*, that is, if there exist for each* ε, $0<\varepsilon<\varepsilon_0$, *functions* $F^{ij}(\varepsilon)=F^{ji}(\varepsilon)\in L^2(\Omega)$ *and there exist functions* $F^{ij}=F^{ji}\in L^2(\Omega)$ *such that* $F^{ij}(\varepsilon)\to F^{ij}$ *in* $L^2(\Omega)$ *as* $\varepsilon\to 0$ *and*

$$\int_{\Gamma^+\cup\Gamma^-}h^{i\pm}v_i\sqrt{g(\varepsilon)}\,\mathrm{d}\Gamma=\int_\Omega F^{ij}(\varepsilon)e_{i\|j}(\varepsilon;\mathbf{v})\sqrt{g(\varepsilon)}\,\mathrm{d}x\quad\textit{for all }0<\varepsilon<\varepsilon_0\textit{ and for all }\mathbf{v}\in\mathbf{V}(\Omega).$$

2. Main results

In what follows we assume that all the assumptions above are satisfied. Let us now improve the second part of the previous theorem and prove that

Theorem 2. *There exist* $\mathbf{u}$ *in* $\mathbf{V}_M^\#(\Omega)$ *and* ζ *in* $\mathbf{V}_M^\#(\omega)$ *such that*

$$\mathbf{u}(\varepsilon)\longrightarrow\mathbf{u}\quad\text{in }\mathbf{V}_M^\#(\Omega)\text{ as }\varepsilon\to 0\quad\text{and}\quad\overline{\mathbf{u}(\varepsilon)}\longrightarrow\zeta\quad\text{in }\mathbf{V}_M^\#(\omega)\text{ as }\varepsilon\to 0$$

and the limit ζ *satisfies the scaled* 2D *variational problem of a linearly elastic generalized membrane shell of the first kind*

$$\mathcal{P}_{\mathcal{M}}^\#(\omega):\quad\begin{cases}\zeta\in\mathbf{V}_M^\#(\omega),\quad\forall\eta\in\mathbf{V}_M^\#(\omega)\\ B_M^\#(\zeta,\eta)=L_M^\#(\eta)\end{cases}$$

if there exist functions $h^{\alpha\beta} = h^{\beta\alpha} \in L^2(\omega)$ *such that*

$$\int_\omega h^i \eta_i \sqrt{a}\,\mathrm{d}y = \int_\omega h^{\alpha\beta}\gamma_{\alpha\beta}(\eta)\sqrt{a}\,\mathrm{d}y \quad \textit{for all } \eta \in \mathbf{V}(\omega). \tag{3}$$

The proof is given for a density applied on the upper surface so that we can identify $h^{i\pm}$ with h^i. The general case is then proved by linearity. In [2], Ph. Ciarlet gave a proof in eleven parts. To prove our theorem, we keep the same pattern of proof but we change the proof or the results of the parts that use the admissibility of the forces, which are parts (ii), (iii), (v) and (vii). The other parts remain unchanged and we use their results when required. For example, to prove part (ii), we admit part (i) and so on.

Let us first remind two useful propositions already proved in [2].

Proposition 1. *We have the following* 3D Inequality of Korn's type: *there exist constants* $C > 0$ *and* $\varepsilon_0 > 0$ *such that*

$$\|\mathbf{v}\|_{1,\Omega} \leqslant \frac{C}{\varepsilon}\left\{\sum_{i,j}\left\|e_{i\|j}(\varepsilon;\mathbf{v})\right\|^2_{0,\Omega}\right\}^{1/2} \quad \textit{for all } \mathbf{v} \in \mathbf{V}(\Omega) \textit{ and all } 0 < \varepsilon < \varepsilon_0. \tag{4}$$

Proposition 2. *If* $w \in L^2(\Omega)$ *satisfies*

$$\int_\Omega w\partial_3 v\,\mathrm{d}x = 0 \quad \textit{for all } v \in H^1(\Omega) \textit{ that vanish on } \Gamma_0,\ \textit{then } w = 0. \tag{5}$$

We now prove two preliminary results that will be used in the proof of Theorem 2.

Lemma 1. *For* $v \in H^1(\Omega)$, *let* $v_{|\Gamma^+}$ *denote the trace of* v *on* Γ^+ *and* $\bar{v}$ *denote the mean value of* v *in the thickness. Then we have*

$$v_{|\Gamma^+} = \bar{v} + \frac{1}{2}\int_{-1}^{1}(1+x_3)\partial_3 v\,dx_3. \tag{6}$$

Proof. The result is obtained after the following integration by parts:

$$\bar{v} = \frac{1}{2}\int_{-1}^{1} v\,\mathrm{d}x_3 = \frac{1}{2}\left(\int_{-1}^{1}\partial_3\big((1+x_3)v\big)\,\mathrm{d}x_3 - \int_{-1}^{1}(1+x_3)\partial_3 v\,\mathrm{d}x_3\right)$$
$$= \frac{1}{2}\left((1+1)v_{|\Gamma^+} - (1-1)v_{|\Gamma^-} - \int_{-1}^{1}(1+x_3)\partial_3 v\,\mathrm{d}x_3\right). \qquad \square$$

Lemma 2. *There exist constants* $c > 0$, $\varepsilon_0 > 0$ *and a function* $G(\varepsilon, x_1, x_2, x_3)$ *such that for all* $0 < \varepsilon < \varepsilon_0$,

$$\sqrt{g(\varepsilon)} = \sqrt{a} + \varepsilon G \quad \textit{with } \|G\|_{0,\infty,\overline{\Omega}} \leqslant c. \tag{7}$$

Proof. In [2], p. 156, it is proved that $g_{\alpha\beta}(\varepsilon) = a_{\alpha\beta} - 2\varepsilon x_3 b_{\alpha\beta} + \mathrm{O}(\varepsilon^2)$. Then, since $g(\varepsilon) = \det((g_{ij}(\varepsilon))_{i,j})$ and $a = \det((a_{\alpha\beta})_{\alpha\beta})$, we have the result by using a first order Taylor development. $\square$

We can now give the proof *of Theorem* 2.

Proof. Let us admit that there exist constants $c_0 > 0$ and $\varepsilon_0 > 0$ such that

$$|\mathbf{v}|^M_\Omega \leqslant c_0\left\{\sum_{i,j}\|e_{i\|j}(\varepsilon;\mathbf{v})\|^2_{0,\Omega}\right\}^{1/2} \quad \text{for all } \mathbf{v} \in \mathbf{V}(\Omega) \text{ and all } 0 < \varepsilon < \varepsilon_0 \tag{8}$$

which is the result of part (i) and let us prove (ii). In order to do that, it is sufficient to prove that there exist constants $c>0$ and $\varepsilon_1>0$ such that for all $0<\varepsilon<\varepsilon_1$,

$$\left|\int_{\Gamma^+} h^{i+}u_i(\varepsilon)\sqrt{g(\varepsilon)}\,\mathrm{d}\Gamma\right| \leqslant c\left\{\sum_{i,j}\|e_{i\|j}(\varepsilon)\|_{0,\Omega}^2\right\}^{1/2}.$$

From (7) we know that

$$\int_{\Gamma^+} h^{i+}u_i(\varepsilon)\sqrt{g(\varepsilon)}\,\mathrm{d}\Gamma = \int_{\Gamma^+} h^{i+}u_i(\varepsilon)\sqrt{a}\,\mathrm{d}\Gamma + \int_{\Gamma^+} h^{i+}u_i(\varepsilon)\varepsilon G\,\mathrm{d}\Gamma. \tag{9}$$

Then, with the help of (3) and (6), we can write the first integral of the rigth side of (9) this way:

$$\int_{\Gamma^+} h^{i+}u_i(\varepsilon)\sqrt{a}\,\mathrm{d}\Gamma = \int_{\omega} h^{\alpha\beta}\gamma_{\alpha\beta}\big(\overline{\mathbf{u}(\varepsilon)}\big)\sqrt{a}\,\mathrm{d}y + \frac{1}{2}\int_{\Omega}(1+x_3)\mathbf{h}\partial_3\mathbf{u}(\varepsilon)\sqrt{a}\,\mathrm{d}x.$$

Hence, applying Cauchy–Schwarz inequality first, using the definition of $|\mathbf{u}(\varepsilon)|_{\Omega}^{M}$ afterwards, and lastly using the majoration (8) we have the following inequalities

$$\begin{aligned}\left|\int_{\Gamma^+} h^{i+}u_i(\varepsilon)\sqrt{a}\,\mathrm{d}\Gamma\right| &\leqslant \big\|h^{\alpha\beta}\sqrt{a}\big\|_{0,\omega}\big\|\gamma_{\alpha\beta}\big(\overline{\mathbf{u}(\varepsilon)}\big)\big\|_{0,\omega} + \frac{1}{2}\big\|(1+x_3)\mathbf{h}\sqrt{a}\big\|_{0,\Omega}\big\|\partial_3\mathbf{u}(\varepsilon)\big\|_{0,\Omega}\\ &\leqslant c\big|\mathbf{u}(\varepsilon)\big|_{\Omega}^{M} \leqslant c\left\{\sum_{i,j}\big\|e_{i\|j}(\varepsilon)\big\|_{0,\Omega}^2\right\}^{1/2}.\end{aligned}$$

At this point, let us insist on the fact that the $h^{\alpha\beta}$'s have to be in $L^2(\omega)$ which can be more restrictive than $\mathbf{h}$ being in the dual of $\mathbf{V}_M^{\#}(\omega)$. That is why, the interesting results obtained by E. Sanchez-Palencia in [7] and [6] about this space are not enough to insure the convergence of $\overline{\mathbf{u}(\varepsilon)}$.

To bound the second integral of the rigth side of (9), we use again the Cauchy–Schwarz inequality, then the continuity of the trace on Γ^+ and the majoration of (7), we conclude with inequality (4). Therefore,

$$\left|\int_{\Gamma^+} h^{i+}u_i(\varepsilon)\varepsilon G\,\mathrm{d}\Gamma\right| \leqslant \varepsilon\|\mathbf{h}^+G\|_{0,\Gamma^+}\big\|\mathbf{u}(\varepsilon)\big\|_{0,\Gamma^+} \leqslant c\varepsilon\big\|\mathbf{u}(\varepsilon)\big\|_{0,\Omega} \leqslant c\varepsilon\big\|\mathbf{u}(\varepsilon)\big\|_{1,\Omega} \leqslant c\left\{\sum_{i,j}\big\|e_{i\|j}(\varepsilon)\big\|_{0,\Omega}^2\right\}^{1/2}.$$

The results of part (iii) are slightly different from those of Ph. Ciarlet. Here, we prove that the limits $e_{i\|j}$ found in part (ii) satisfy

$$e_{1\|3}=0,\quad e_{2\|3}=0\quad\text{and}\quad e_{3\|3}=-\frac{\lambda}{\lambda+2\mu}a^{\alpha\beta}e_{\alpha\|\beta}.$$

In order to do so, in $\mathcal{P}(\varepsilon;\Omega)$ we let $\mathbf{v}:=\varepsilon\mathbf{w}$, $\mathbf{w}$ being an arbitrary function in the space $\mathbf{V}(\Omega)$, and we let ε approach zero; we obtain the equation

$$\int_{\Omega}\big\{2\mu a^{\alpha\sigma}e_{\sigma\|3}\partial_3 w_\alpha + \big(\lambda a^{\sigma\tau}e_{\sigma\|\tau} + (\lambda+2\mu)e_{3\|3}\partial_3 w_3\big)\big\}\sqrt{a}\,\mathrm{d}x = 0,$$

which, combined with (5), implies the result. Now let us prove part (v), which has become,

$$\int_{\omega} a^{\alpha\beta\sigma\tau}\overline{e_{\sigma\|\tau}}\gamma_{\alpha\beta}(\eta)\sqrt{a}\,\mathrm{d}y = \int_{\omega} h^{\alpha\beta}\gamma_{\alpha\beta}(\eta)\sqrt{a}\,\mathrm{d}y\quad\text{for all }\eta\in\mathbf{V}(\omega).$$

We just need to check that

$$\int_{\omega} h^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{v})\sqrt{a}\,\mathrm{d}y = \lim_{\varepsilon\to 0}\int_{\Gamma^+} h^{i+}v_i\sqrt{g(\varepsilon)}\,\mathrm{d}\Gamma\quad\text{for all }\mathbf{v}\in\mathbf{V}(\Omega)\text{ independent of the transverse variable.}$$

A function $\mathbf{v}\in\mathbf{V}(\Omega)$ independent of the transverse variable x_3 satisfies $\partial_3\mathbf{v}=\mathbf{0}$. That is why, using the same decomposition as in the proof of part (ii), we have

$$\int_{\Gamma^+} h^{i+}v_i\sqrt{g(\varepsilon)}\,\mathrm{d}\Gamma=\int_{\omega} h^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{v})\sqrt{a}\,\mathrm{d}y+\varepsilon\int_{\Gamma^+} h^{i+}v_i G\,\mathrm{d}\Gamma$$

and the expected result when we let $\varepsilon\to 0$. To prove part (vii), we use the following strong and weak convergences proved in parts (vi), (iv) and (ii):

$$\varepsilon\mathbf{u}(\varepsilon)\longrightarrow\mathbf{0}\quad\text{and}\quad\partial_3\mathbf{u}(\varepsilon)\longrightarrow\mathbf{0}\quad\text{in }\mathbf{L}^2(\Omega)\text{ as }\varepsilon\to 0,\quad\text{and,}\quad\gamma_{\alpha\beta}\big(\overline{\mathbf{u}(\varepsilon)}\big)\rightharpoonup\overline{e_{\alpha\|\beta}}\quad\text{in }\mathbf{L}^2(\omega)\text{ as }\varepsilon\to 0.\tag{10}$$

To prove part (vii), we only need to prove the following result

$$\lim_{\varepsilon\to 0}\int_{\Gamma^+} h^{i+}u_i(\varepsilon)\sqrt{g(\varepsilon)}\,\mathrm{d}\Gamma=\int_{\omega} h^{\alpha\beta}\overline{e_{\alpha\|\beta}}\sqrt{a}\,\mathrm{d}y.$$

From the proof of part (ii), we know that

$$\int_{\Gamma^+} h^{i+}u_i(\varepsilon)\sqrt{g(\varepsilon)}\,\mathrm{d}\Gamma=\int_{\omega} h^{\alpha\beta}\gamma_{\alpha\beta}\big(\overline{\mathbf{u}(\varepsilon)}\big)\sqrt{a}\,\mathrm{d}y+\frac{1}{2}\int_{\Omega}(1+x_3)\mathbf{h}\partial_3\mathbf{u}(\varepsilon)\sqrt{a}\,\mathrm{d}x+\int_{\Gamma^+} h^{i+}\varepsilon u_i(\varepsilon)G\,\mathrm{d}\Gamma\tag{11}$$

and because of (10) we have

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int_{\Omega}(1+x_3)\mathbf{h}\partial_3\mathbf{u}(\varepsilon)\sqrt{a}\,\mathrm{d}x=0,\quad \lim_{\varepsilon\to 0}\int_{\omega} h^{\alpha\beta}\gamma_{\alpha\beta}\big(\overline{\mathbf{u}(\varepsilon)}\big)\sqrt{a}\,\mathrm{d}y=\int_{\omega} h^{\alpha\beta}\overline{e_{\alpha\|\beta}}\sqrt{a}\,\mathrm{d}y$$

$$\text{and}\quad\lim_{\varepsilon\to 0}\int_{\Gamma^+} h^{i+}\varepsilon u_i(\varepsilon)G\,\mathrm{d}\Gamma=0.$$

Then, to get the announced result, we just have to let $\varepsilon\to 0$ in (11). □

Provided we suppose more regularity on $h^{\alpha\beta}$ and possibly impose boundary conditions, we can rewrite (3) this way

$$\int_{\omega} h^i\eta_i\sqrt{a}\,\mathrm{d}y=\int_{\omega}\chi^i\big(\big(h^{\alpha\beta}\big)_{\alpha\beta}\big)\eta_i\,\mathrm{d}y\quad\text{for all }\eta\in\mathbf{V}(\omega),$$

where $\chi^\sigma((h^{\alpha\beta})_{\alpha\beta}):=-\partial_\sigma(h^{\sigma\alpha}\sqrt{a})-\Gamma^\sigma_{\alpha\beta}h^{\alpha\beta}$ and $\chi^3((h^{\alpha\beta})_{\alpha\beta}):=-b_{\alpha\beta}h^{\alpha\beta}$. That leads to the following ill-posed coupled edp system

$$\chi^i\big(\big(h^{\alpha\beta}\big)_{\alpha\beta}\big)=h^i\sqrt{a}\quad\text{in }\omega\text{ and }h^{\alpha\beta}=0\text{ on }\partial\omega\setminus\gamma_0.$$

The next step consists in finding conditions on h^i to make sure that there is at least one solution to the previous system. This was done in the case of a particular parabolic geometry in Poutous [5].

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