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Group Theory

On minimal non-(torsion-by-nilpotent) and non-((locally finite)-by-nilpotent) groups

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Abstract

Let Ω be a class of groups. A group is said to be minimal non-Ω if it is not an Ω-group, while all its proper subgroups belong to Ω. In this Note we prove that a minimal non-(torsion-by-nilpotent) (respectively, non-((locally finite)-by-nilpotent)) group G is a finitely generated perfect group which has no proper subgroup of finite index and such that $G/Frat(G)$ is an infinite simple group, where $Frat(G)$ stands for the Frattini subgroup of G. ***To cite this article: N. Trabelsi, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Sur les groupes minimaux non-(périodiques-par-nilpotents) et non-((localement finis)-par-nilpotents). Soit Ω une classe de groupes. Un groupe est dit minimal non-Ω s'il n'est pas un Ω-groupe alors que tous ses sous-groupes propres le sont. Dans cette Note, nous prouvons que si G est un groupe minimal non-(périodique-par-nilpotent) (respectivement, non-((localement fini)-par-nilpotent)), alors G est un groupe parfait de type fini qui n'admet pas de sous-groupe propre d'indice fini et tel que $G/Frat(G)$ est un groupe simple infini, où $Frat(G)$ désigne le sous-groupe de Frattini de G. ***Pour citer cet article : N. Trabelsi, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit Ω une classe de groupes. Un groupe G est dit minimal non-Ω s'il n'est pas un Ω-groupe alors que tous ces sous-groupes propres le sont. L'étude des groupes minimaux non-Ω, pour diverses classes de groupes Ω, a fait l'objet de nombreuses publications (voir par exemple, [1–4,6,8] et [10]). En particulier, dans [4] (respectivement, [10]) l'étude des groupes G minimaux non-$\mathcal{N}$ (respectivement, non-$\mathcal{FN}$) est menée et il est prouvé, parmi de nombreux résultats, que si G est infini et de type fini, alors $G/Frat(G)$ est un groupe simple infini, où $\mathcal{N}$ (respectivement, $\mathcal{F}$) désigne la classe de tous les groupes nilpotents (respectivement, finis) et $Frat(G)$ est le sous-groupe de Frattini de G. Dans ce qui suit, on obtient un résultat analogue sur les groupes minimaux non-$\mathcal{XN}$, dans les cas où $\mathcal{X}$ désigne la classe des groupes périodiques ou la classe des groupes localement finis. Plus précisément, on prouvera le résultat suivant :

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Théorème 0.1. *Soient $c \geqslant 0$ un entier, $\mathcal{N}_c$ la classe des groupes nilpotents de classe égale au plus à c et $\mathcal{X}$ la classe des groupes de torsion ou la classe des groupes localement finis. Si G est un groupe minimal non-$\mathcal{XN}$ (respectivement, non-$\mathcal{XN}_c$), alors G est un groupe parfait de type fini qui n'admet pas de sous-groupe propre d'indice fini et tel que $G/Frat(G)$ est un groupe simple infini.*

Signalons que les groupes minimaux non-$\mathcal{XN}$ (respectivement non-$\mathcal{XN}_c$) existent. En effet Ol'shanskii [5] a construit un groupe simple, de type fini et sans torsion, dont tous les sous-groupes propres sont cycliques.

1. Introduction

Let Ω be a class of groups. A group is said to be minimal non-Ω if it is not a Ω-group, while all its proper subgroups belong to Ω. Many results have been obtained on minimal non-Ω, for various classes of groups Ω (see for example, [1–4,6,8] and [10]). In particular, in [4] (respectively, [10]) it is proved, among many results, that if G is an infinite finitely generated minimal non-$\mathcal{N}$ (respectively, non-$\mathcal{FN}$) group, then $G/Frat(G)$ is an infinite simple group, where $\mathcal{N}$ (respectively, $\mathcal{F}$) denotes the class of nilpotent (respectively, finite) groups and $Frat(G)$ is the Frattini subgroup of G. In this note we obtain a similar result on minimal non-$\mathcal{XN}$ groups, where $\mathcal{X}$ stands for the class of torsion groups or the class of locally finite groups. More precisely, we shall prove the following result:

Theorem 1.1. *Let $c \geqslant 0$ be an integer. Denote by $\mathcal{N}_c$ the class of nilpotent groups of class at most c and by $\mathcal{X}$ the class of torsion groups or the class of locally finite groups. If G is a minimal non-$\mathcal{XN}$ (respectively, non-$\mathcal{XN}_c$) group, then G is a finitely generated perfect group which has no proper subgroup of finite index and such that $G/Frat(G)$ is an infinite simple group.*

Note that minimal non-$\mathcal{XN}$ (respectively, non-$\mathcal{XN}_c$) groups exist. Indeed, the group constructed by Ol'shanskii [5] is an infinite simple torsion-free finitely generated group whose proper subgroups are cyclic.

2. Proof of Theorem 1.1

Our first lemma is an immediate consequence of [9, Lemma 2.1] but we give a proof to keep our Note reasonably self contained.

Lemma 2.1. *Let G be a group whose proper subgroups are in the class $\mathcal{XN}$. Then G belongs to $\mathcal{XN}$ if it satisfies one of the following two conditions*:

(i) *G is finitely generated and has a proper subgroup of finite index,*
(ii) *G is not finitely generated.*

Proof. (i) Suppose that G is finitely generated and let H be a proper normal subgroup of finite index in G. Then H belongs to $\mathcal{XN}$ and it is finitely generated. It follows that $\gamma_{k+1}(H)$ is in $\mathcal{X}$ for some integer $k \geqslant 0$. Since H is of finite index in G, $G/\gamma_{k+1}(H)$ is a finitely generated group in the class $\mathcal{NF}$, so that $G/\gamma_{k+1}(H)$ satisfies the maximal condition on subgroups. It follows that every $\mathcal{X}$-subgroup of $G/\gamma_{k+1}(H)$ is finite. Consequently, every proper subgroup of $G/\gamma_{k+1}(H)$ is finite-by-nilpotent. Now Lemma 4 of [2] states that a finitely generated locally graded group with finite-by-nilpotent proper subgroups is itself finite-by-nilpotent. Since $G/\gamma_{k+1}(H)$ is clearly locally graded, we deduce that $G/\gamma_{k+1}(H)$ is finite-by-nilpotent, so that G belongs to $\mathcal{XN}$.

(ii) Suppose now that G is not finitely generated and let x, y be two elements of finite order in G. The subgroup $\langle x, y\rangle$, being proper in G, is in $\mathcal{XN}$. Thus xy^{-1} is of finite order, so G has a torsion subgroup T. As G is not finitely generated, T is locally in $\mathcal{XN}$, so that T belongs to $\mathcal{X}$, since it is periodic. If G/T is not finitely generated, then it is locally in $\mathcal{XN}$; and since G/T is torsion-free, it is locally nilpotent and its proper subgroups are nilpotent. Now Theorem 2.1 of [8] states that a torsion-free locally nilpotent group with proper nilpotent subgroups is itself nilpotent. Therefore G/T is nilpotent, so that G is an $\mathcal{XN}$-group. Now if G/T is finitely generated, then there exists a finitely generated subgroup X such that $G = XT$. Since G is not finitely generated, X is proper in G, so that X belongs to $\mathcal{XN}$. We deduce that G/T is in $\mathcal{XN}$, so that G/T is nilpotent because it is torsion-free. Therefore, G belongs to $\mathcal{XN}$. □

Since finitely generated locally graded groups have proper subgroups of finite index, the previous lemma admits the following consequence:

Corollary 2.2. *Let G be a locally graded group whose proper subgroups are in the class $\mathcal{XN}$. Then G belongs to $\mathcal{XN}$.*

Lemma 2.3. *Let G be a group whose proper subgroups are in the class $\mathcal{XN}$. If G is not perfect, then G belongs to $\mathcal{XN}$.*

Proof. Since G' is a proper subgroup, it is in the class $\mathcal{XN}$. So G belongs to $\mathcal{X}(\mathcal{NA})$, where $\mathcal{A}$ denotes the class of Abelian groups. Therefore there exists a normal $\mathcal{X}$-subgroup F such that G/F is soluble. By Corollary 2.2, G/F belongs to $\mathcal{XN}$, so that G is an $\mathcal{XN}$-group, as claimed. □

Lemma 2.4. *Let G be a group whose proper subgroups are in the class $\mathcal{XN}$ and let N be a proper normal subgroup of G. If G/N' is an $\mathcal{XN}$ group, then G belongs to $\mathcal{XN}$.*

Proof. Since G/N' belongs to $\mathcal{XN}$, there is an integer $i \geqslant 0$ such that $\gamma_{i+1}(G/N')$ is an $\mathcal{X}$-group. Clearly from Lemma 2.3, we can assume that $G' = G$, so that $(G/N')' = G/N'$. Thus $G/N' = \gamma_{i+1}(G/N')$ and hence G/N' belongs to $\mathcal{X}$. On the other hand N, being proper, belongs to the class $\mathcal{XN}$. Therefore there exists an integer $k \geqslant 0$ such that $\gamma_{k+1}(N)$ is in $\mathcal{X}$. If $k = 0$ then N is an $\mathcal{X}$-group. Since G/N' belongs to $\mathcal{X}$, we deduce that G is in $\mathcal{X}$. Thus we can suppose that $k > 0$ and hence $N' \geqslant \gamma_{k+1}(N)$. Factoring G by $\gamma_{k+1}(N)$, we may assume that N is nilpotent. Let T be the torsion subgroup of N. Then T is an $\mathcal{X}$-group, so we may further assume without loss of generality that N is torsion-free. Let x be an element of $Z_2(N)$. By considering the homomorphism $f: g \mapsto [g, x]$ from N into $Z(N)$, we see that $N' \leqslant \ker f$, thus $N/\ker f$ is an $\mathcal{X}$-group and this implies that $\operatorname{Im} f = [N, x]$ is in $\mathcal{X}$. Hence $[N, x] = 1$ as N is torsion-free. This means that x is an element of $Z(N)$, hence $Z(N) = Z_2(N) = N$, so that $N' = 1$. Since G/N' belongs to $\mathcal{X}$, we obtain that G is in $\mathcal{X}$ and, a fortiori, G is in $\mathcal{XN}$, as required. □

Lemma 2.5. *Let A and F be two subgroups of a group G such that A is normal and Abelian, F is an $\mathcal{X}$-group and $G = AF$. If every proper subgroup of G belongs to $\mathcal{XN}$, then G is in $\mathcal{XN}$.*

Proof. Let H be a proper subgroup of G. Then H is in the class $\mathcal{XN}$ and therefore it has a torsion subgroup T which belongs to $\mathcal{X}$. Hence H/T is a torsion-free nilpotent group belonging to the class $\mathcal{AX}$. By Lemma 6.33 of [7], it follows that H/T is Abelian. So that H belongs to $\mathcal{XA}$ and therefore every proper subgroup of G is in $\mathcal{XA}$. Factoring G by the torsion subgroup of A, we may assume that A is torsion-free. Clearly, from Lemma 2.3, we may further assume that G is perfect. Let x be an element in G; then $\langle A, x\rangle$ is a proper subgroup of G. So that $\langle A, x\rangle$ belongs to $\mathcal{XA}$. It follows that $[A, x]$ is an $\mathcal{X}$-group, hence $[A, x] = 1$ as A is normal and torsion-free. Therefore A is central in G and hence $G' = F'$ is an $\mathcal{X}$-group. This gives that G belongs to $\mathcal{XA}$, and consequently G is in $\mathcal{XN}$, as claimed. □

Lemma 2.6. *Let M and N be two proper subgroups of a group G such that N is normal and $G = MN$. If every proper subgroup of G belongs to $\mathcal{XN}$, then G is in $\mathcal{XN}$.*

Proof. Clearly we can assume from Lemma 2.3 that G is perfect. Since M is proper in G, it belongs to $\mathcal{XN}$. Hence $\gamma_{k+1}(M)$ is an $\mathcal{X}$-group for some integer $k \geqslant 0$. By using an induction on k, we can see that $\gamma_{k+1}(G) = \gamma_{k+1}(MN) \leqslant \gamma_{k+1}(M)N$. For we have that $\gamma_{k+1}(G) = \gamma_{k+1}(MN)$ is generated by commutators of the form $w = [h_1, \ldots, h_{k+1}]$, where each h_i belongs to MN. Put $h_{k+1} = xy$ with x in M and y in N, then $w = [h_1, \ldots, h_k, y][h_1, \ldots, h_k, x][h_1, \ldots, h_k, x, y]$ is a product of 3 commutators say w_1, w_2 and w_3 respectively. Since N is normal in G, w_1 and w_3 are in N. Now using the inductive hypothesis we have that $w_2 = [uz, x]$ for some u in $\gamma_k(M)$ and some z in N. So that $w_2 = [u, x][u, x, z][z, x]$ is an element of $\gamma_{k+1}(M)N$, hence w belongs to $\gamma_{k+1}(M)N$ and therefore $\gamma_{k+1}(MN) \leqslant \gamma_{k+1}(M)N$, as claimed. Now $\gamma_{k+1}(G) = G$ as $G = G'$, so that $G = \gamma_{k+1}(M)N$, hence

$G/N' = (N/N')(\gamma_{k+1}(M)N'/N')$. Since $\gamma_{k+1}(M)$ is in $\mathcal{X}$, we have also that $(\gamma_{k+1}(M)N'/N')$ is an $\mathcal{X}$-group. By Lemma 2.5, it follows that G/N' is in $\mathcal{XN}$ and therefore Lemma 2.4 permits us to conclude that G belongs to $\mathcal{XN}$, as claimed. □

From the previous lemma, we can deduce the following result:

Corollary 2.7. *If G is a minimal non-$\mathcal{XN}$ group, then every pair of proper normal subgroups generates a proper subgroup. Moreover, every proper normal subgroup N of G is omissible*; *that is, $HN = G$ implies $H = G$ for every subgroup H of G.*

Proof of Theorem 1.1. (i) Suppose first that G is a minimal non-$\mathcal{XN}$ group. From Lemma 2.1 and Lemma 2.3, G is a finitely generated perfect group which has no proper subgroup of finite index. So $G/Frat(G)$ is infinite. Suppose that $G/Frat(G)$ is not simple and let N be a normal subgroup of G such that $Frat(G) \lneq N \lneq G$. Therefore there is a maximal subgroup M of G such that $N \nleq M$. It follows that $G = MN$. We deduce, by Corollary 2.7, that $G = M$, which is a contradiction. Therefore $G/Frat(G)$ is simple.

(ii) Suppose now that G is a minimal non-$\mathcal{XN}_c$ group. If G is an $\mathcal{XN}$-group, then its torsion subgroup T belongs to $\mathcal{X}$ and G/T is a torsion-free nilpotent group. But a well known result of Zaicev [11] states that an infinite nilpotent group whose proper subgroups are in $\mathcal{N}_c$ is itself in the class $\mathcal{N}_c$. Thus G/T is in $\mathcal{N}_c$ and hence G belongs to $\mathcal{XN}_c$, which is a contradiction. So that G is a minimal non-$\mathcal{XN}$ group. It follows from (i) that G is a finitely generated perfect group which has no proper subgroup of finite index and such that $G/Frat(G)$ is an infinite simple group. □

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