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Partial Differential Equations

Results dealing with the behavior of the integrated density of states of random divergence operators

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Abstract

In this Note we generalize and improve results proven for acoustic operators given by Najar in 2003. It deals with the behavior of the integrated density of states of random divergence operators of the form $H_\omega = \sum_{i,j=1}^{d} \partial_{x_i} a_{i,j}(\omega, x)\partial_{x_j}$ on the internal band edges of the spectrum. ***To cite this article: H. Najar, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Des résultats sur le comportement de la densité d'états integrèe de l'opérateur de divergence aléatoire. Dans cette Note on généralise et en améliore des résultats prouvés pour les opérateurs acoustique par Najar (2003). Il concerne le comportement de la densité d'états intégrée de l'opérateur de divergence aléatoire ayant la forme $H_\omega = \sum_{i,j=1}^{d} \partial_{x_i} a_{i,j}(\omega, x)\partial_{x_j}$ aux bords internes du spectre. ***Pour citer cet article : H. Najar, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

On considère l'opérateur de divergence aléatoire de la forme

$$H_\omega = -\nabla A_\omega^{-1} \nabla,$$

avec

$$A_\omega = A_0(x) + \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma B(x - \gamma),$$

où

– $A_0 : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ est $\mathbb{Z}^d$-périodique et uniformément elliptique i.e. il existe $C > 0$ tel que, $\forall x \in \mathbb{R}^d$,

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$$\frac{1}{C}\cdot I_d \leqslant A_0(x) \leqslant C\cdot I_d;$$

- $B:\mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ est continuement différentiable vérifiant
  (i) soit $\exists \nu > d+2$ et $\exists C > 0$ tel que
  $$\frac{1}{C}\mathbf{1}_{|x|\leqslant 1/C} I_d \leqslant B(x) \leqslant C\big(1+|x|\big)^{-\nu} I_d;$$
  (ii) soit $\exists \nu \in (d, d+2]$ et $\exists C > 0$ tel que
  $$\frac{1}{C}\big(1+|x|\big)^{-\nu} I_d \leqslant B(x) \leqslant C\big(1+|x|\big)^{-\nu} I_d;$$
- $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ est une famille de variables aléatoires non constantes, indépendantes et identiquement distribuées prenant des valeurs dans [0, 1]. On suppose que
  $$\lim_{\varepsilon\to 0^+} \frac{\log|\log \mathbb{P}(\{\omega_0 \in (1-\varepsilon, 1]\})|}{\log\varepsilon} = -\kappa, \quad \kappa \in [0, +\infty[. \tag{1}$$

L'objectif de cette Note est de donner le comportement de la densité d'états intégrée aux bords des lacunes internes du spectre de H_ω. On distingue entre les cas (i) et (ii). On démontre qu'il y a deux régimes possibles de comportement, classique et quantique. La valeur du paramètre κ dans (1) est à l'origine de la transition entre ces deux régimes.

1. Introduction

We consider the random divergence operator

$$H_\omega = -\nabla A_\omega^{-1}\nabla = \sum_{i,j=1}^{d} \partial_{x_i} a_{i,j}(\omega, x)\partial_{x_j}; \tag{2}$$

where A_ω is an elliptic, $d\times d$-matrix valued, $\mathbb{Z}^d$-ergodic random field. i.e there exists some constant $\rho_* > 1$, satisfying

$$\frac{1}{\rho_*}|\xi|^2 \leqslant \langle A_\omega\xi, \xi\rangle \leqslant \rho_*|\xi|^2, \quad \forall \xi \in \mathbb{C}^d. \tag{3}$$

This operator describes a vibrating membrane in the random medium as well as in the particular case when $A_\omega = \varrho_\omega \cdot I_d$ (I_d is the identity matrix and ϱ_ω is a real function) we get the acoustic operator [2,9,10]. The interest of this operator both from the physical and the mathematical point of view is known [14].

We denote by $H_{\omega,\Lambda}$ the restriction of H_ω to Λ with self-adjoint boundary conditions. As H_ω is elliptic, the resolvent of $H_{\omega,\Lambda}$ is compact and, consequently, the spectrum of $H_{\omega,\Lambda}$ is discrete and made of isolated eigenvalues of finite multiplicity [12]. We define

$$N_\Lambda(E) = \frac{1}{\operatorname{vol}(\Lambda)}\cdot \#\{\text{eigenvalues of } H_{\omega,\Lambda} \leqslant E\}. \tag{4}$$

Here $\operatorname{vol}(\Lambda)$ is the volume of Λ in the Lebesgue sense and $\#E$ is the cardinal of E.

It is shown that the limit of $N_\Lambda(E)$ when Λ tends to $\mathbb{R}^d$ exists almost surely and is independent of the boundary conditions. It is called the *integrated density of states* of H_ω (IDS as an acronym). See [6].

1.1. The model

We consider that A_ω has an Anderson form i.e.

$$A_\omega = A_0(x) + \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma B(x-\gamma);$$

where
(A.0)

- $A_0:\mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$, $\mathbb{Z}^d$-periodic and uniformly elliptic i.e. there exists $C > 0$ such that, $\forall x \in \mathbb{R}^d$,

$$\frac{1}{C}\cdot I_d \leqslant A_0(x) \leqslant C\cdot I_d.$$

- $B:\mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ continuously differentiable such that for $C_{0,k}$, the box of side length $2k+1$ and has 0 as a center, there exists $0 \leqslant g_- \in L^2(C_{0,0})$ (g_- non-identically zero) such that for all $\gamma \in \mathbb{Z}^d$ and a.e. $x \in C_{0,0}$ we have,
  (i) either $\exists \nu > d+2$ and $\exists C > 0$ such that
  $$g_-(x-\gamma)I_d \leqslant B(x-\gamma) \leqslant \frac{C}{(1+|\gamma|)^\nu} I_d;$$
  (ii) or $\exists \nu \in (d, d+2]$ and $\exists C > 0$ such that
  $$\frac{1}{C}\big(1+|\gamma|\big)^{-\nu} I_d \leqslant B(x-\gamma) \leqslant C\cdot\big(1+|\gamma|\big)^{-\nu} I_d.$$
  We assume also that in the two cases we have
  $$\big\|{}^t\nabla \cdot B(x)\big\| \leqslant C\cdot\big(1+|x|\big)^{-\nu-1}.$$
- $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a family of random variables independently and identically distributed taking values [0, 1]. We suppose that the common probability measure is supported in [0, 1], and
  $$\lim_{\varepsilon\to 0^+} \frac{\log|\log \mathbb{P}(\{\omega_0 \in (1-\varepsilon, 1]\})|}{\log \varepsilon} = -\kappa, \quad \kappa \in [0, +\infty[. \tag{5}$$

By this, H_ω is a measurable family of essentially self-adjoint and ergodic operators on $L^2(\mathbb{R}^d)$ with a domain $\mathcal{D}(H_\omega) = C_0^\infty(\mathbb{R}^d)$ [2,14].

1.2. The main assumptions

It is convenient to consider H_ω as a perturbation of some periodic operator. For this we set,

$$H_1 = -\nabla A_1^{-1}\nabla \quad \text{with } A_1 = A_0 + \sum_{\gamma\in\mathbb{Z}^d} B(\cdot - \gamma).$$

By this

$$H_\omega = H_1 + \Delta H_\omega$$

with

$$\Delta H_\omega = H_\omega - H_1 = -\nabla\bigg(A_\omega^{-1}\bigg(\sum_{\gamma\in\mathbb{Z}^d}\overline{\omega}_\gamma B(x-\gamma)\bigg)A_1^{-1}\bigg)\nabla \geqslant 0,$$

$$\langle \Delta H_\omega u, u\rangle = \sum_{\gamma\in\mathbb{Z}^d}\overline{\omega}_\gamma \big\langle B(x-\gamma)A_1^{-1}\nabla, A_\omega^{-1}\nabla u\big\rangle; \quad \forall u \in \mathcal{D}(H_\omega).$$

Here for any $\gamma \in \mathbb{Z}^d$ we set $\overline{\omega}_\gamma = (1-\omega_\gamma)$ and the positivity of ΔH_ω is a consequence of the fact that $A_\omega \leqslant A_1$. We notice that H_1 is a $\mathbb{Z}^d$-periodic operator. We denote by n its IDS.

As we study the internal Lifshitz tails it is natural to assume that H_1 has a spectral gap below E_+. More precisely we assume that:

(**A.1**) There exists E_+ and $\delta > 0$ such that $\sigma(H_1) \cap [E_+, E_+ + \delta) = [E_+, E_+ + \delta)$ and $\sigma(H_1) \cap (E_+ - \delta, E_+] = \emptyset$.
As, $\Delta H_\omega \geqslant 0$, the spectrum Σ of H_ω contains an interval of the form $[E_+, E_+ + a]$ ($a > 0$) [5]. As we are interested in the behavior of the IDS in the neighborhood of E_+, we require that E_+ remains always the edge of a gap for Σ, one requires that the following assumption holds.

(**A.2**) There exists $\delta > 0$ such that $\Sigma \cap [E_+ - \delta, E_+) = \emptyset$.

Remark 1. In [3] the existence of open spectral gaps in the spectrum of certain periodic acoustic operators for $d = 2$ and 3 is studied and in [4] the band-gap structure of the spectrum of the elliptic operators in divergence form for $d \geqslant 2$ is considered.

By adding a disorder parameter g in the equation which defines A_ω i.e. $A_\omega = A_0 + g\sum_{\gamma\in\mathbb{Z}^d}\omega_\gamma B(\cdot-\gamma)$, we can choose g small enough so that the spectral gap in $\sigma(H_1)$ will not be closed after the perturbation [2].

2. Results and discussions

The main result of this Note is the following:

Theorem 2.1. *Let H_ω be the operator defined by* (2). *We assume that* **(A.1)** *and* (**A.2**) *hold. Then if*

1. *B is of short range type then,*

$$\lim_{\varepsilon\to 0^+}\frac{\log(n(E_++\varepsilon)-n(E_+))}{\log\varepsilon}=\frac{d}{2}\Leftrightarrow \lim_{\varepsilon\to 0^+}\frac{\log|\log(N(E_++\varepsilon)-N(E_+))|}{\log\varepsilon}=-\left(\frac{d}{2}+\kappa\right). \tag{6}$$

2. *B is of long range type then,*

$$\lim_{\varepsilon\to 0^+}\frac{\log(n(E_++\varepsilon)-n(E_+))}{\log\varepsilon}=\frac{d}{2}\Rightarrow \lim_{\varepsilon\to 0^+}\frac{\log|\log(N(E_++\varepsilon)-N(E_+))|}{\log\varepsilon}$$
$$=-\sup\left(\frac{d}{2}+\kappa,\ \frac{d}{\nu-d}\right). \tag{7}$$

If $\kappa+\frac{d}{2}<\frac{d}{\nu-d}$ then,

$$\lim_{\varepsilon\to 0^+}\frac{\log|\log(N(E_++\varepsilon)-N(E_+))|}{\log\varepsilon}=-\frac{d}{\nu-d}. \tag{8}$$

Now, let us comment the result. One notices that the behavior of the random variables is linked up to the Lifshitz exponent, and determines if one is located in a classical regime or in a quantum one; i.e. if the kinetic energy intervenes or if it does not in the Lifshitz exponent. In the long range case, one sees that it depends on the value of κ, the Lifshitz asymptotics are classical (for $\kappa<\frac{d}{\nu-d}-\frac{d}{2}$) or quantum (for $\kappa>\frac{d}{\nu-d}-\frac{d}{2}$). In other terms, in the case of the long range potential, the Lifshitz exponent depends on the uncertainty principle, i.e. on the kinetic energy only in the case when ($\frac{d}{\nu-d}<\kappa+\frac{d}{2}$). In contrast, when ($\frac{d}{\nu-d}>\kappa+\frac{d}{2}$), then the Lifshitz asymptotics are not governed by the same considerations. This is due to the fact that in the long range case as the potential decreases slowly and locally, the potential is an empirical average of random variables. This leads to the fact that its effect is more important and more influencing than the spatial extension of the considered state.

From what has been said previously, one concludes that the value of κ is responsible for the transition between those two regimes.

The proof of the main result is now classic and based on the technique of periodic approximations which where originally stated by Klopp in [7]. It is quite close and follows the same steps used in [9,10] and [8] from which this work is inspired. Giving the main changes, we omit details and we refer the reader to the above references.

2.1. The lower bound

By assumption, there is a spectral gap below E_+ of length at least $\delta>0$. Thus, for $\varepsilon<\delta$ we have

$$N(E_++\varepsilon)-N(E_+)=N(E_++\varepsilon)-N(E_+-\varepsilon).$$

By this, it suffices to lower bound $N(E_++\varepsilon)-N(E_+-\varepsilon)$. For N being large, we will show that H_{ω,Λ_N} (H_{ω,Λ_N} is H_ω restricted to Λ_N with Dirichlet boundary conditions) has a large number of eigenvalues in $[E_+-\varepsilon, E_++\varepsilon]$ with a large probability. For this, we will construct a family of approximate eigenvectors associated to approximate eigenvalues of H_{ω,Λ_N} in $[E_+-\varepsilon, E_++\varepsilon]$. These functions can be constructed from a Floquet eigenvector $\varphi(x,\theta)$ of H_1 associated with E_+. Locating this eigenvector in θ and imposing to $\overline{\omega}_\gamma$ to be small for γ in some well chosen cube, one obtains an approximate eigenfunction of H_{ω,Λ_N}. Locating the eigenfunction in x in several disjointed places, we get several eigenfunctions, two by two orthogonal. The subtlety is in the good choice of the size of the cube. Using the same computation done in [8–10] we get that, in the case of the short range we have to estimate the double logarithm of:

$$\mathbb{P}\biggl(\biggl\{\omega;\ \sum_{\gamma\in\Lambda_\alpha(\varepsilon^s)}\overline{\omega}_\gamma\bigl(1+|\gamma|\bigr)^{-\nu}\leqslant\frac{\varepsilon^{1+\alpha}}{2}\biggr\}\biggr)\geqslant\mathbb{P}\biggl\{\omega;\ \forall\gamma\in\Lambda_\alpha(\varepsilon^s);\overline{\omega}_\gamma\leqslant\frac{\varepsilon^{1+\alpha}}{C}\biggr\}=\biggl(\mathbb{P}\biggl\{\overline{\omega}_0\leqslant\frac{\varepsilon^{1+\alpha}}{C}\biggr\}\biggr)^{\sharp\Lambda_\alpha(\varepsilon^s)};$$

here $1>\alpha>0$; $\Lambda_\alpha(\zeta)=\{\gamma\in\mathbb{Z}^d;\ \forall 1\leqslant j\leqslant d;\ |\gamma_j|\leqslant\zeta^{-(\frac{1}{2}+\alpha)}\}$ and $s<1$ resp. $s=1$ if n is non-degenerate resp. degenerate.

In the long range case, we deal with $\mathbb{P}_{\varepsilon,\alpha,1}=\mathbb{P}(\{\omega;\ |\beta|\leqslant\varepsilon^{-(1+\alpha)/2};\ \sum_{\gamma\in\mathbb{Z}^d}\overline{\omega}_\gamma(1+|\beta-\gamma|)^{-\nu}\leqslant\varepsilon^{1+\alpha}\})$. First we notice that $\mathbb{P}_{\varepsilon,\alpha,1}\geqslant\mathbb{P}_2\cdot\mathbb{P}_1$, where

$$\mathbb{P}_1=\mathbb{P}\bigl\{\omega;\ \forall\gamma\text{ such that }|\gamma|\leqslant\varepsilon^{-(1-\alpha)/2},\ \overline{\omega}_\gamma\leqslant\varepsilon^{1+\alpha}\bigr\},$$

and

$$\mathbb{P}_2=\mathbb{P}\bigl\{\omega;\ \forall\gamma\text{ such that }\varepsilon^{-(1-\alpha)/2}<|\gamma|\leqslant\varepsilon^{-(1+2\alpha)/(\nu-d)},\overline{\omega}_\gamma\leqslant\varepsilon^{1+\alpha}\bigl(1+\operatorname{dist}(\gamma,C_{0,\varepsilon^{-(1-\alpha)/2}})\bigr)^{(\nu-d)(1-\alpha)}\bigr\}.$$

For α and ε small enough we get that

$$\log\mathbb{P}_{\varepsilon,\alpha,1}\geqslant-\varepsilon^{-(\kappa+d/2)(1+\alpha)}-\varepsilon^{-\kappa(1+\alpha)}\sum_{\varepsilon^{-(1-\alpha)/2}\leqslant|\gamma|\leqslant\varepsilon^{-(1+2\alpha)/(\nu-d)}}\bigl(1+\operatorname{dist}(\gamma,C_{0,\varepsilon^{-(1-\alpha)/2}})\bigr)^{-\kappa(\nu-d)(1-\alpha)}.\tag{9}$$

Since, if $(\nu-d)\kappa>d$ the last sum converges when α is chosen small enough such that $(1-\alpha)(\nu-d)\kappa>d$, we get,

$$\liminf_{\varepsilon\to0^+}\frac{\log|\log(\mathbb{P}_{\varepsilon,\alpha,1})|}{\log\varepsilon}\geqslant-(1+\alpha)\biggl(\kappa+\frac{d}{2}\biggr).\tag{10}$$

In the case when $(\nu-d)\kappa<d$, for ε being small, one computes the sum in (9) and gets the following estimation

$$\sum_{\varepsilon^{-(1-\alpha)/2}\leqslant|\gamma|\leqslant\varepsilon^{-(1+2\alpha)/(\nu-d)}}\bigl(1+\operatorname{dist}(\gamma,C_{0,\varepsilon^{-(1-\alpha)/2}})\bigr)^{-\kappa(\nu-d)(1-\alpha)}\leqslant C\cdot\varepsilon^{\kappa(1-\alpha)}\cdot\varepsilon^{-d(1+\alpha)/(\nu-d)}.\tag{11}$$

2.2. *The upper bound*

First, we notice that the short range case is given in [9]. See parts III. B and V. A of [9] for the reduction to the discrete model.

For the long range case, when $\frac{d}{\nu-d}>\kappa+\frac{d}{2}$, we notice that we have no assumption made on the behavior of n, the IDS of the periodic operator. The proof goes exactly as the one given in [10] for $\kappa=0$.

When $\frac{d}{\nu-d}<\kappa+\frac{d}{2}$, using a similar result to Theorem 3.2 of [9] we get that for an energy E close to E_+, $N(E)-N(E_+)$ can be upper bounded by $N_{\mathcal{E}_0}(C\cdot(E-E_+)+E_+)$, the IDS of the bounded random operator $H_\omega^0=\Pi_0H_\omega\Pi_0$. Here Π_0, is the spectral projection for H_1 on the band starting at E_+. Then we get that H_ω^0 is equivalent to a random Jacobi matrix and having E_+ as a spectrum bottom. We recall that in this case one supposes that n is non-degenerate. It is proven in [7] that this is equivalent to say that the Floquet eigenvalues of H_1 reaching the band edge E_+ has only non-degenerate quadratic extrima at that edge. Lemma 5.5 of [9] can be extended to the present case using the properties of A_ω and B.

By this, using the discrete Fourier transformation and following [8,11,13] we get that it suffices to estimate N^a, the IDS of the following Anderson discrete operator acting on $l^2(\mathbb{Z}^d)$:

$$\bigl(H_\omega^au\bigr)(\alpha)=E_+\cdot u(\alpha)+\sum_{|\alpha-\beta|=1}\bigl(u(\alpha)-u(\beta)\bigr)+\bigl(V_\omega^au\bigr)(\alpha).\tag{12}$$

Here V_ω^a the diagonal infinite matrix with $v_\alpha(\omega)=\sum_{\beta\in\mathbb{Z}^d}\overline{\omega}_\beta(1+|\alpha-\beta|)^{-\nu}$ for the αth diagonal coefficient.

For $u\in l^2(\mathbb{Z}^d\cap C_{0,k})$, let H_1^k, V_ω^k and H_ω^k be the following discrete operators

$$\bigl(H_1^ku\bigr)(\alpha)=E_+\cdot u(\alpha)+\sum_{|\alpha-\beta|=1,\ \beta\in C_{0,k}}\bigl(u(\alpha)-u(\beta)\bigr),\qquad\bigl(V_\omega^ku\bigr)(\alpha)=v_\alpha(\omega)u(\alpha)$$

and

$$H_\omega^k = H_0^k + V_\omega^k.$$

Let N_k^a be the IDS of H_ω^k. From [11,13], we know that for a good choice of k, the IDS at energy E is quite well approximated by the probability to find a state energy less than E. Precisely we have the following relation:

$$N^a(E) \leqslant N_k^a(E) \leqslant C \cdot \mathbb{P}_k(E) = \mathbb{P}\big(\big\{H_\omega^k \text{ admits at least an eigenvalue less than } E\big\}\big).$$

The estimation of the last probability is based on probabilistic arguments [1] and given in [8].

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