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Mathematical Analysis

# Supremum over inverse image of functions in the Bloch space

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**Abstract**

We will prove that for certain classes of functions $f$ in the $\alpha$-Bloch space $\mathcal{B}^\alpha$ such that $f(0)=0$, the $\mathcal{B}^\alpha$ norm is obtained taking supremum over $f^{-1}(\Sigma_\varepsilon)$, where $\Sigma_\varepsilon=\{z\colon |\arg z|<\varepsilon\}$. ***To cite this article: J.C. Ramos Fernández, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***


**Résumé**

**La borne supérieure de l'image inverse de fonctions dans l'espace de Bloch.** Nous démontrerons que pour certaines classes de fonctions $f$ dans l'espace $\alpha$-Bloch $\mathcal{B}^\alpha$ et telles que $f(0)=0$, la norme $\mathcal{B}^\alpha$ s'obtient comme la borne supérieure sur $f^{-1}(\Sigma_\varepsilon)$, où $\Sigma_\varepsilon=\{z\colon |\arg z|<\varepsilon\}$. ***Pour citer cet article : J.C. Ramos Fernández, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***


## 1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane. A function $f$ is called *$\alpha$-Bloch function* if it is analytic on $\mathbb{D}$ and $\|f\|_\alpha=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|<\infty$. This defines a seminorm, and the $\alpha$-Bloch functions form a complex Banach space $\mathcal{B}^\alpha$ with the norm $\|f\|_{\mathcal{B}^\alpha}=|f(0)|+\|f\|_\alpha$.

When $\alpha=1$ we get back the classical Bloch space $\mathcal{B}$ and it is known that $\mathcal{B}$ is conformally invariant in the sense that if $a\in\mathbb{D}$, then $\|f\circ\varphi_a\|_1=\|f\|_1$, where $\varphi_a$ is a Möbius transformation from the unit disk onto itself; that is, $\varphi_a(z)=(a-z)/(1-\bar{a}z)$, $z\in\mathbb{D}$.

In this Note, we are interested in knowing if, given $\varepsilon>0$ and $\alpha>-1$, we can find a constant $\delta>0$, depending only on $\alpha$ and $\varepsilon$, such that

$$\sup_{z\in f^{-1}(\Sigma_\varepsilon)}\big(1-|z|^2\big)^\alpha\big|f'(z)\big|\geqslant\delta\|f\|_\alpha, \tag{1}$$

for all functions $f\in\mathcal{B}^\alpha$ satisfying $f(0)=0$, where $\Sigma_\varepsilon=\{w\in\mathbb{C}\colon |\arg(w)|<\varepsilon\}$.

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This question was suggested by an article of D. Marshall and W. Smith [3] where they analyze a problem of this type for functions in the classic Bergman's space (without weight) $A^p$. The principal result of [3, Theorem 1.1] ensures that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_{f^{-1}(\Sigma_\varepsilon)} \left|f(z)\right| \mathrm{d}A(z) > \delta \int_{\mathbb{D}} \left|f(z)\right| \mathrm{d}A(z), \tag{2}$$

for any univalent function in $A^1$ fixing the origin, where $\mathrm{d}A(z)$ is the two-dimensional Lebesgue measure on $\mathbb{D}$. It is an open problem to show whether their result still holds if the hypothesis that $f$ is univalent were omitted.

Pérez-González and Ramos [4] have extended the results of Marshall and Smith to the widest class of weighted Bergman space $A^p_\alpha$ when $\alpha > 2p-1$ and $p > 1$. Recently, in [1] the authors showed that an estimate like (2) is not possible for the Besov space $B_p = B_p(\mathbb{D})$, with $p > 1$, where an holomorphic functions $f$ on $\mathbb{D}$ belongs to $B_p$ if

$$\|f\|^p_{B_p} = \int_{\mathbb{D}} \left(1-|z|^2\right)^{p-2} \left|f'(z)\right|^p \mathrm{d}A(z) < \infty.$$

However, they showed that for any $p > 1$ and $\varepsilon > 0$, there exists a constant $K > 0$ depending only on $p$ such that

$$\int_{f^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right)^{p-2} \left|f'(z)\right|^p \mathrm{d}A(z) \geqslant K(p)\varepsilon \frac{|f'(0)|^{p+4}}{\|f\|^4_{B_p}},$$

for any nonnull function $f \in B_p$ with $f(0) = 0$.

In this Note, we get a similar result for the $\alpha$-Bloch space. We will show that if $\alpha = 1$, then the inequality (1) is not true, for all $\varepsilon > 0$ and for all nonnull $f \in \mathcal{B}^\alpha$ with $f(0) = 0$. However we will prove the following result:

**Theorem 1.1.** *Assume $\alpha > -1$. Then there exists a constant $K(\alpha) > 0$ depending only on $\alpha$ such that*

$$\sup_{z \in f^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| \geqslant K(\alpha)\varepsilon \frac{|f'(0)|^4}{\|f\|^3_\alpha}, \tag{3}$$

*for any nonconstant function $f \in \mathcal{B}^\alpha$ with $f(0) = 0$.*

The above result implies that the inequality (1) is true for all $\varepsilon > 0$ and $\alpha \geqslant 3$ if we consider univalent functions in the $\alpha$-Bloch space fixing the origin. This extends the result of Marshall and Smith in [3].

## 2. Failure for $\alpha = 1$

In this section we will show that the inequality (1) is not true, for all $\varepsilon > 0$ and for all nonnull $f \in \mathcal{B}^\alpha$ with $f(0) = 0$ when $\alpha = 1$. We can note that if $\alpha = 1$, then $\|\cdot\|_\alpha = \|\cdot\|_1$ is the Möbius invariant seminorm. Thus for each $f \in \mathcal{B}$ and any $a \in \mathbb{D}$ we have $\|f\|_1 = \|g_a\|_1$, where $g_a = f \circ \varphi_a - f(a)$. So $g_a(0) = 0$ and a change of variables shows

$$\sup_{z \in g_a^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right) \left|g_a'(z)\right| = \sup_{w \in f^{-1}(\Sigma_\varepsilon + f(a))} \left(1-|w|^2\right) \left|f'(w)\right|.$$

So if (1) held it would follow that $\delta\|f\|_1 = \delta\|g_a\|_1 < \sup_{w \in f^{-1}(\Sigma_\varepsilon + f(a))}(1-|w|^2)|f'(w)|$, for all $a \in \mathbb{D}$ and $f \in \mathcal{B}$, which is not possible.

## 3. Proof of the main theorem

The following well known result (see [2,5]) will play an important role in the proof of the main theorem. We will denote by $D(a, r)$ the Euclidean disk with center $a$ and radius $r$.

**Theorem 3.1** *(1/4-Koebe). If $g(0) = 0$ and $g'(0) = 1$, then $D(0, \frac{1}{4}) \subset \Omega$, where $g$ is a conformal map from the unit disk $\mathbb{D}$ into a domain $\Omega$.*

We now give the proof of our main theorem. We can observe that inequality (3) is true if $f'(0)=0$ therefore we can suppose that $f'(0)\neq 0$. Since $f$ is an analytic function on the disk $D(0,\frac{1}{2})$, we can apply the Cauchy integral formula to obtain

$$\left|f'(z)\right| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{|f'(r\mathrm{e}^{\mathrm{i}\theta})|}{|r\mathrm{e}^{\mathrm{i}\theta}-z|} r\,\mathrm{d}\theta,$$

for any $|z|<\frac{1}{2}$ and any $\frac{1}{2}<r<1$. Integrating from $r=\frac{3}{4}$ to $r=\frac{7}{8}$ we have

$$\frac{1}{8}\left|f'(z)\right| \leqslant 2 \int\limits_{\{3/4<|s|<7/8\}} \left|f'(s)\right| \mathrm{d}A(s), \tag{4}$$

for any $|z|<\frac{1}{2}$. Also, we can note that there exist positive constants $K_1(\alpha)$, $K_2(\alpha)$ such that $K_1(\alpha)\leqslant (1-|s|^2)^\alpha \leqslant K_2(\alpha)$, for all $\frac{3}{4}<|s|<\frac{7}{8}$. Thus, substituting in (4), we can see that there exists a constant $K_3(\alpha)>0$ such that

$$\left|f'(z)\right| \leqslant K_3(\alpha)\|f\|_\alpha, \tag{5}$$

for all $z$ such that $|z|<\frac{1}{2}$.

Next, we define the function $h(w)=\frac{1}{2K_3(\alpha)\|f\|_\alpha}\{f'(\frac{w}{2})-f'(0)\}$, for $w\in\mathbb{D}$. It is not hard to see that $h$ is analytic function on $\mathbb{D}$ satisfying $h(0)=0$ and $|h(w)|\leqslant 1$. Invoking Schwarz's lemma, we obtain $|h(w)|\leqslant |w|$, for all $w\in\mathbb{D}$. Hence, we have

$$\left|f'(z)-f'(0)\right| \leqslant 4K_3(\alpha)\|f\|_\alpha |z|, \tag{6}$$

for all $|z|<\frac{1}{2}$. Now, if we take $R=\frac{1}{8K_3\|f\|_\alpha}|f'(0)|$, then from (5) we have $R<\frac{1}{8}$ and from (6) we obtain that $|f'(z)-f'(0)|<\frac{1}{2}|f'(0)|$, for all $|z|<R$. This implies that the function $f$ is one to one on the disk $D(0,R)$. Thus, if we define the function $g(z)=\frac{1}{Rf'(0)}f(Rz)$, $z\in\mathbb{D}$, we can see that $g(0)=0$ and $g'(0)=1$. Then by the 1/4-Koebe theorem, we have $D(0,\frac{1}{4})\subset g(\mathbb{D})$. This implies that $D(0,\sigma)\subset f(D(0,R))$, where

$$\sigma = \frac{|f'(0)|^2}{32K_3\|f\|_\alpha}.$$

Therefore

$$\begin{aligned}
\sup_{z\in f^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| &\geqslant \int\limits_{f^{-1}(\Sigma_\varepsilon\cap D(0,\sigma))\cap D(0,R)} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| \mathrm{d}A(z) \\
&\geqslant \frac{K_1(\alpha)}{K_3(\alpha)\|f\|_\alpha} \int\limits_{f^{-1}(\Sigma_\varepsilon\cap D(0,\sigma))\cap D(0,R)} \left|f'(z)\right|^2 \mathrm{d}A(z) \\
&= K_4(\alpha)\varepsilon \frac{|f'(0)|^4}{\|f\|_\alpha^3},
\end{aligned}$$

where in the second inequality we have used $|z|<R<\frac{1}{8}$, $|f'(z)|\leqslant K_3\|f\|_\alpha$, and also, the fact that if $f$ is 1–1 on the set $E$ then

$$\int\limits_E \left|f'(z)\right|^2 \mathrm{d}A(z) = \operatorname{area}\bigl(f(E)\bigr).$$

This concludes the proof of the theorem. □

Now we can extend the result of Marshall and Smith in [3] to the $\alpha$-Bloch spaces.

**Corollary 3.2.** *If $\alpha\geqslant 3$, then for all $\varepsilon>0$, there exists a constant $\delta>0$, depending only on $\alpha$ and $\varepsilon$, such that*

$$\sup_{f^{-1}(\Sigma_\varepsilon)} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| \geqslant \delta\|f\|_\alpha, \tag{7}$$

*for any univalent function $f\in\mathcal{B}^\alpha$ with $f(0)=0$.*

**Proof.** Indeed, it is known (see [2, Distortion Theorem]) that if $f$ is a conformal map from the unit disk $\mathbb{D}$ into a domain $\Omega$ and $f(0)=0$, then for all $z\in\mathbb{D}$ holds

$$\frac{1-|z|}{(1+|z|)^3}\left|f'(0)\right|\leqslant\left|f'(z)\right|\leqslant\frac{1+|z|}{(1-|z|)^3}\left|f'(0)\right|.$$

In particular, for all $z\in\mathbb{D}$ and $\alpha\geqslant 3$, we have $|f'(0)|\geqslant\frac{1}{16}(1-|z|^2)^3|f'(z)|\geqslant\frac{1}{16}(1-|z|^2)^\alpha|f'(z)|$, and $|f'(0)|\geqslant\frac{1}{16}\|f\|_\alpha$. Thus using the estimate (3) we have

$$\sup_{z\in f^{-1}(\Sigma_\varepsilon)}\left(1-|z|^2\right)^\alpha\left|f'(z)\right|\geqslant\frac{K(\alpha)}{(16)^4}\varepsilon\|f\|_\alpha.$$

This complete the proof of Corollary 3.2. □

**Remark 1.** We finish this Note with two questions: If we omit the condition that the function is univalent in Corollary 3.2, does the result continue to be true? Is the inequality (1) true for all $\varepsilon>0$ when $\alpha\in(1,3)$?

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