Algebraic geometry

Cohomology of the Hilbert scheme of points on a surface with values in the double tensor power of a tautological bundle

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Abstract

Let X be a smooth quasi-projective complex algebraic surface and $X^{[n]}$ be the Hilbert scheme of n points on X. We compute the cohomology $H^*(X^{[n]}, E^{[n]} \otimes E^{[n]})$ of the Hilbert scheme $X^{[n]}$ with values in the double tensor power of the tautological bundle $E^{[n]}$, associated to a line bundle E on the surface X. ***To cite this article: L. Scala, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***
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Résumé

Cohomologie du schéma de Hilbert ponctuel d'une surface à valeurs dans la puissance tensorielle double d'un fibré tautologique. Soient X une surface algébrique complexe quasi-projective lisse et $X^{[n]}$ le schéma de Hilbert ponctuel sur X. On calcule la cohomologie $H^*(X^{[n]}, E^{[n]} \otimes E^{[n]})$ du schéma de Hilbert $X^{[n]}$ à valeurs dans la puissance tensorielle double du fibré tautologique $E^{[n]}$, associé à un fibré en droites E sur la surface X. ***Pour citer cet article : L. Scala, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***
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Soit M une variété algébrique complexe quasi-projective lisse et G un groupe fini d'automorphismes de M. La correspondance de McKay se fonde sur le principe que, en étudiant les objets G-équivariants sur M, on peut obtenir des informations sur une éventuelle résolution crépante Y du quotient M/G. Plus précisément, le théorème de Bridgeland–King–Reid [1] affirme premièrement que, sous certaines hypothèses techniques, une telle résolution crépante existe et peut être réalisée par la composante irréductible Y du G-schéma de Hilbert $G \operatorname{Hilb}(M)$ contenant les orbites libres. Deuxièmement, il exprime la correspondance de McKay avec une équivalence de catégories dérivées entre la catégorie dérivée des faisceaux cohérents sur Y et la catégorie dérivée des faisceaux cohérents G-équivariants sur M. Cette équivalence est réalisée via une transformation de Fourier–Mukai ayant par noyau la famille universelle $\mathcal{Z} \subseteq Y \times M$.

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Soit maintenant X une surface algébrique complexe quasi-projective lisse. Le travail de Haiman sur le schéma de Hilbert isospectral [4] montre que, dans le cas de l'action du groupe symétrique $\mathfrak{S}_n$ sur le produit X^n de la surface X, la composante irréductible Y du $\mathfrak{S}_n$-schéma de Hilbert $\mathfrak{S}_n \operatorname{Hilb}(X^n)$ de X^n contenant les orbites libres peut être identifiée au schéma de Hilbert ponctuel $X^{[n]}$ sur la surface X ; dans cette identification la famille universelle $\mathcal{Z}$ correspond au schéma de Hilbert isospectral $B^n \subseteq X^{[n]} \times X^n$. De plus, les hypothèses du théorème de Bridgeland–King–Reid sont vérifiées. Par conséquent, on obtient une équivalence de catégories dérivées $\boldsymbol{\Phi}$ entre $\mathbf{D}^b(X^{[n]})$ et $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$ via la transformation de Fourier–Mukai ayant par noyau le schéma de Hilbert isospectral.

Soit maintenant E un fibré en droites sur la surface X et soit $E^{[n]}$ le fibré tautologique sur le schéma de Hilbert $X^{[n]}$ naturellement associé au fibré en droites E. On utilise le travail de Haiman afin de reconduire le calcul de la cohomologie $H^*(X^{[n]}, (E^{[n]})^{\otimes k})$ du schéma de Hilbert à valeurs dans la puissance tensorielle k-ième du fibré tautologique au calcul de l'hypercohomologie $\mathfrak{S}_n$-équivariante de X^n à valeurs dans l'image $\boldsymbol{\Phi}((E^{[n]})^{\otimes k})$ de $(E^{[n]})^{\otimes k}$ par la transformée de Bridgeland–King–Reid $\boldsymbol{\Phi}$.

L'image $\boldsymbol{\Phi}((E^{[n]})^{\otimes k})$ de la puissance tensorielle du fibré tautologique par la transformation de Bridgeland–King–Reid résulte être une composition de transformations de Fourier–Mukai. D'après le travail de Haiman sur les polygraphes [5], le noyau de la transformation de Fourier–Mukai composée est le polygraphe $D(n,k)$.

Pour $k=1$, le faisceau structural du polygraphe admet une résolution $\mathfrak{S}_n$-équivariante en termes de faisceaux structuraux de ses composantes et de leurs intersections. Ceci permet d'introduire sur X^n un complexe borné $\mathfrak{S}_n$-équivariant $\mathcal{C}_E^\bullet$, de type Čech, exact en degré > 0 et quasi-isomorphe à l'image $\boldsymbol{\Phi}(E^{[n]})$. Dans le cas général la structure du polygraphe est plus compliquée. On a quand même un morphisme naturel : $\mathcal{C}_E^\bullet \otimes^L \cdots \otimes^L \mathcal{C}_E^\bullet \to \boldsymbol{\Phi}((E^{[n]})^{\otimes k})$ dans la catégorie dérivée $\mathfrak{S}_n$-équivariante $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$. On en déduit que ce morphisme est un quasi-isomorphisme en degré > 0 ; en degré 0 c'est un épimorphisme dont le noyau est le sous-faisceau de torsion. En particulier, la cohomologie $H^q(\boldsymbol{\Phi}((E^{[n]})^{\otimes k}))$ du complexe image est zero pour $q > 0$: l'image voulue est alors concentrée en degré zéro, donc quasi-isomorphe à un faisceau. L'examen de la suite spectrale hyperdérivée associée au produit tensoriel dérivé k-ième $\mathcal{C}_E^\bullet \otimes^L \cdots \otimes^L \mathcal{C}_E^\bullet$ permet d'interpréter le faisceau $\boldsymbol{\Phi}((E^{[n]})^{\otimes k})$ comme terme $E_\infty^{0,0}$.

Le calcul du terme $E_\infty^{0,0}$ est compliqué en général. Cependent, sa connaissance, bien que de grand intérêt, n'est pas essentielle afin de calculer la cohomologie voulue : tout ce dont on a besoin est la connaissance des $\mathfrak{S}_n$-invariants $\boldsymbol{\Phi}((E^{[n]})^{\otimes k})^{\mathfrak{S}_n}$ de l'image $\boldsymbol{\Phi}((E^{[n]})^{\otimes k})$. Maintenant le groupe symétrique $\mathfrak{S}_n$ opère sur la suite spectrale hyperdérivée. La suite spectrale des invariants $\mathcal{E}_1^{p,q} := (E_1^{p,q})^{\mathfrak{S}_n}$ de la suite $E_1^{p,q}$ par le groupe symétrique permet le calcul des invariants $\boldsymbol{\Phi}((E^{[n]})^{\otimes k})^{\mathfrak{S}_n}$ ou, en d'autres termes, de l'image directe $\mathbf{R}\mu_*((E^{[n]})^{\otimes k})$ par le morphisme de Hilbert–Chow : le calcul du terme $\mathcal{E}_\infty^{0,0}$ est maintenant beaucoup plus accessible, au moins pour $k=2$. Ceci est suffisant afin d'obtenir le Théorème 1.1, qui donne la cohomologie $H^*(X^{[n]}, E^{[n]} \otimes E^{[n]})$ en termes des cohomologies de X à valeurs dans $E^{\otimes 2}$, E et dans son faisceau structural. On obtient comme corollaire la cohomologie des puissances symétriques et extérieures doubles $S^2 E^{[n]}$ et $\Lambda^2 E^{[n]}$.

1. Statement of the results

Let X be a smooth quasi-projective complex algebraic surface and E a line bundle on X. Let $X^{[n]}$ be the Hilbert scheme of n points on X and $E^{[n]}$ the tautological bundle on $X^{[n]}$ naturally associated to the line bundle E on X. We relate the cohomology of $X^{[n]}$ with values in the double tensor power $(E^{[n]})^{\otimes 2}$ of the tautological bundle with the cohomologies of X with values in $E^{\otimes 2}$, E and its structural sheaf.

Theorem 1.1. *Let X be a smooth quasi-projective complex algebraic surface and E a line bundle on X. The cohomology $H^*(X^{[n]}, E^{[n]} \otimes E^{[n]})$ of the Hilbert scheme $X^{[n]}$ of n points on X with values in the double tensor power of the tautological bundle $E^{[n]}$ is given by the isomorphism of graded modules*:

$$H^*\big(X^{[n]}, E^{[n]} \otimes E^{[n]}\big) \simeq H^*(X, E)^{\otimes 2} \otimes S^{n-2} H^*(X, \mathcal{O}_X) \oplus H^*(X, E^{\otimes 2}) \otimes \mathcal{J}$$

where $\mathcal{J}$ is the ideal of classes in $S^{n-1} H^*(X, \mathcal{O}_X)$ vanishing on the scheme $\{x\} \times S^{n-2} X$, with x a fixed point in X.

As a corollary, we obtain the cohomology of the double symmetric and exterior powers of $E^{[n]}$:

$$H^*\big(X^{[n]}, S^2 E^{[n]}\big) \simeq S^2 H^*(X, E) \otimes S^{n-2} H^*(X, \mathcal{O}_X) \oplus H^*\big(X, E^{\otimes 2}\big) \otimes \mathcal{J}$$

$$H^*\big(X^{[n]}, \Lambda^2 E^{[n]}\big) \simeq \Lambda^2 H^*(X, E) \otimes S^{n-2} H^*(X, \mathcal{O}_X).$$

The formula for $S^2 E^{[n]}$ had already been obtained by Danila [3] for the case $n \leqslant 3$. Proofs of the results will appear in a forthcoming article.

2. The Bridgeland–King–Reid theorem and the isospectral Hilbert scheme

The main idea in order to prove Theorem 1.1 is to use recent results in McKay correspondence by Bridgeland, King and Reid, adapted by Haiman for the case of the Hilbert scheme. Let M be a smooth quasi-projective algebraic variety and let $G \subseteq \mathrm{Aut}(M)$ be a finite group of automorphisms of M. We recall that the G-Hilbert scheme $G\,\mathrm{Hilb}(M)$ parametrizes G-invariant subschemes Z of M of length $|G|$ such that $H^0(\mathcal{O}_Z) \simeq \mathbb{C}[G]$ (see [1], [6]). There is a G-Hilbert–Chow morphism: $\mu : G\,\mathrm{Hilb}(M) \to M/G$, surjective and birational on the irreducible component Y of $G\,\mathrm{Hilb}(M)$ containing free orbits. Let $\mathcal{Z} \subseteq Y \times M$ be the universal family. We have the diagram:

$$
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\ p\ } & M \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\
Y & \xrightarrow{\ \mu\ } & M/G
\end{array}
$$

Theorem 2.1 *(Bridgeland–King–Reid [1]). Let M be a smooth quasi-projective complex algebraic variety, G a finite subgroup of $\mathrm{Aut}(M)$, Y the irreducible component of $G\,\mathrm{Hilb}(M)$ containing free orbits and $\mathcal{Z} \subseteq Y \times M$ the universal family. Suppose that ω_M is locally trivial as G-sheaf and that $\dim Y \times_{M/G} Y \leqslant \dim Y + 1$. Then Y is a crepant resolution of M/G and the Fourier–Mukai functor:*

$$\Phi := \mathbf{R}p_* \circ q^* : \mathbf{D}^b(Y) \longrightarrow \mathbf{D}_G^b(M) \tag{1}$$

is an equivalence of categories.

The application of the Bridgeland–King–Reid theorem in the case of the action of the symmetric group $\mathfrak{S}_n$ on the product variety X^n, where X is a smooth quasi-projective surface, is made possible by the work by Haiman on the isospectral Hilbert scheme [4].

Definition 2.2. The *isospectral Hilbert scheme* B^n is the reduced fiber product $B^n := (X^{[n]} \times_{S^n X} X^n)_{\mathrm{red}}$ of X^n and $X^{[n]}$ over the symmetric product $S^n X$.

Haiman gives a characterization of B^n as the blow-up of X^n along the scheme-theoretic union of all pairwise diagonals $\bigcup_{i<j} \Delta_{ij}$. Let p and q be the projections of B^n onto X^n and $X^{[n]}$, respectively. Haiman proves that the isospectral Hilbert scheme B^n is normal, Cohen–Macauley and Gorenstein. This implies that the finite morphism $q : B^n \to X^{[n]}$ is flat of degree n, hence realizing B^n as a flat family over $X^{[n]}$ of $\mathfrak{S}_n$-invariant subschemes of X^n and allowing the comparison and the identification between $X^{[n]}$ and the component $Y \subseteq \mathfrak{S}_n\,\mathrm{Hilb}(X^n)$ and between B^n and $\mathcal{Z}$. From now on, we set $M = X^n$, $G = \mathfrak{S}_n$, $Y = X^{[n]}$, $\mathcal{Z} = B^n$ and μ the Hilbert–Chow morphism. The hypothesis of the Bridgeland–King–Reid theorem are satisfied, since $S^n X$ is Gorenstein and the Hilbert–Chow morphism is a semismall resolution.

Theorem 2.3 *(Haiman). Let X a smooth quasi-projective algebraic surface. The Hilbert scheme $X^{[n]}$ of n points on X identifies to the irreducible component Y of the $\mathfrak{S}_n$-Hilbert scheme $\mathfrak{S}_n\,\mathrm{Hilb}(X^n)$ containing free orbits. In this identification the isospectral Hilbert scheme B^n corresponds to the universal family $\mathcal{Z}$ over Y. Moreover, we have an equivalence of derived categories*:

$$\boldsymbol{\Phi} := \mathbf{R}p_* \circ q^* : \mathbf{D}^b\big(X^{[n]}\big) \longrightarrow \mathbf{D}_{\mathfrak{S}_n}^b\big(X^n\big). \tag{2}$$

Consequently the cohomology of the Hilbert scheme $X^{[n]}$ with values in an arbitrary coherent sheaf F can be now computed as the $\mathfrak{S}_n$-equivariant hypercohomology of X^n with values in the image $\boldsymbol{\Phi}(F)$ by the BKR equivalence or as the hypercohomology of $S^n X$ with values in the invariants $\boldsymbol{\Phi}(F)^{\mathfrak{S}_n}$:

$$H^*\big(X^{[n]}, F\big) \simeq \mathbb{H}_{\mathfrak{S}_n}^*\big(X^n, \boldsymbol{\Phi}(F)\big) \simeq \mathbb{H}^*\big(S^n X, \boldsymbol{\Phi}(F)^{\mathfrak{S}_n}\big). \tag{3}$$

3. Tensor power of a tautological bundle and polygraphs

Let us define the tautological bundle $E^{[n]}$ associated to a line bundle E on the surface X. For brevity's sake, if $\mathcal{X}$ and $\mathcal{Y}$ are two smooth algebraic varieties and $\mathcal{W} \subseteq \mathcal{X} \times \mathcal{Y}$ is a subscheme of their product, we will indicate with $\Phi^{\mathcal{W}}_{\mathcal{X} \to \mathcal{Y}}$ the Fourier–Mukai functor $\mathbf{R}p_{\mathcal{Y}*}(\mathcal{O}_{\mathcal{W}} \otimes^L p^*_{\mathcal{X}}(-))$.

Definition 3.1. Let $\Xi \subseteq X^{[n]} \times X$ be the universal family for the Hilbert scheme and let p_X and $p_{X^{[n]}}$ be the projections of Ξ onto X and onto $X^{[n]}$, respectively. Let E be a line bundle on X. The tautological bundle on $X^{[n]}$ associated to the line bundle E on X is the rank n vector bundle:

$$E^{[n]} := p_{X^{[n]}*} p^*_X E.$$

We can also say that the tautological bundle $E^{[n]}$ is obtained by applying to E the Fourier–Mukai functor $\Phi^{\Xi}_{X \to X^{[n]}}$ from X to $X^{[n]}$ with kernel Ξ. In order to express the tensor power $(E^{[n]})^{\otimes k}$ as well in terms of Fourier–Mukai functors, let $\Xi(n, k)$ be the k-fold fiber product of the universal family Ξ over the Hilbert scheme $X^{[n]}$:

$$\Xi(n, k) := \underbrace{\Xi \times_{X^{[n]}} \cdots \times_{X^{[n]}} \Xi}_{k\text{-times}}.$$

The scheme $\Xi(n, k)$ is naturally a subscheme of $X^{[n]} \times X^k$. Let $p_{X^{[n]}}$ and p_{X^k} be its projections onto $X^{[n]}$ and X^k, respectively. The k-fold tensor power of the tautological bundle can be expressed as:

$$\left(E^{[n]}\right)^{\otimes k} = p_{X^{[n]}*} p^*_{X^k} E^{\boxtimes k}$$

or as the image of $E^{\boxtimes k}$ on X^k for the Fourier–Mukai transform $\Phi^{\Xi(n,k)}_{X^k \to X^{[n]}}$.

In order to compute the cohomology of $X^{[n]}$ with values in the k-fold tensor power $(E^{[n]})^{\otimes k}$ of the tautological bundle $E^{[n]}$, it suffices by (3) to compute the $\mathfrak{S}_n$-equivariant cohomology of X^n with values in the image $\boldsymbol{\Phi}((E^{[n]})^{\otimes k})$ via the BKR transform. The complex $\boldsymbol{\Phi}((E^{[n]})^{\otimes k})$ turns out to be the image of $E^{\boxtimes k}$ for the composition of Fourier–Mukai functors:

$$\boldsymbol{\Phi}\left(\left(E^{[n]}\right)^{\otimes k}\right) = \Phi^{B_n}_{X^{[n]} \to X^n} \circ \Phi^{\Xi(n,k)}_{X^k \to X^{[n]}}\left(E^{\boxtimes k}\right). \tag{4}$$

It is well known that the composition of two Fourier–Mukai functors is a third Fourier–Mukai functor; Haiman [5] explicitly computed the kernel of the resulting composition in (4) in terms of the polygraph $D(n, k)$, which we now define. Let D be the subscheme of $X^n \times X$ defined as the scheme-theoretic union $D = \bigcup_{i=1}^n D_i$, where $D_i = \Delta_{i,n+1}$ are pairwise diagonals in $X^{n+1} = X^n \times X$.

Definition 3.2. The *polygraph* $D(n, k)$ is the *reduced* k-fold fiber product of D over X^n:

$$D(n, k) := \left(\underbrace{D \times_{X^n} \cdots \times_{X^n} D}_{k\text{-times}}\right)_{\text{red}}. \tag{5}$$

Haiman's results on polygraphs [5] yield the following:

Proposition 3.3. *The composition of Fourier–Mukai functors in* (4) *is given by the Fourier–Mukai functor* $\Phi^{D(n,k)}_{X^k \to X^n}$ *from X^k to X^n with kernel the polygraph $D(n, k)$:*

$$\Phi^{B_n}_{X^{[n]} \to X^n} \circ \Phi^{\Xi(n,k)}_{X^k \to X^{[n]}} = \Phi^{D(n,k)}_{X^k \to X^n}.$$

As a consequence we can express the image of the tensor power $(E^{[n]})^{\otimes k}$ for the BKR transform as:

$$\boldsymbol{\Phi}\left(\left(E^{[n]}\right)^{\otimes k}\right) = \Phi^{D(n,k)}_{X^k \to X^n}\left(E^{\boxtimes k}\right).$$

4. The image of the tensor product of the tautological bundle for the BKR transform

4.1. The case $k = 1$

In the case $k = 1$ we explicitly computed the image above. In this case the structural sheaf of the polygraph $D(n, 1) = D$ admits a $\mathfrak{S}_n$-equivariant Čech-type resolution in terms of structural sheaves of the partial diagonals D_i and their intersections. This fact allows to find a nice $\mathfrak{S}_n$-equivariant resolution of the image $\boldsymbol{\Phi}(E^{[n]})$. Let $I \subseteq \{1, \ldots, n\}$, $I \neq \emptyset$ be a multi-index. Let $p_I : X^n \to X^I$ be the projection onto the factors indexed by I and let $i_I : X \to X^I$ be the diagonal immersion. Let E_I be the sheaf on X^n defined by: $E_I := p_I^* i_{I*} E$, supported on the partial diagonal Δ_I. Consider the complex $(\mathcal{C}_E^p, \partial_E^p)$ defined by:

$$\mathcal{C}_E^p := \bigoplus_{|I|=p+1} E_I, \qquad \partial_E^p(f)_I = \sum_{i \in I} (-1)^{\sharp\{b \in I, \, b < i\}} f_{I \setminus \{i\}}\big|_{\Delta_I}. \tag{6}$$

This complex possesses a natural $\mathfrak{S}_n$-action making the differentials ∂_E^i equivariant: in other words $\mathcal{C}_E^\bullet \in \mathbf{D}_{\mathfrak{S}_n}^b(X^n)$. Moreover it turns out to be exact in degree > 0.

Theorem 4.1. *The image $\boldsymbol{\Phi}(E^{[n]})$ of the tautological bundle $E^{[n]}$ associated to a line bundle E on X for the BKR transform is isomorphic in the equivariant derived category $\mathbf{D}_{\mathfrak{S}_n}^b(X^n)$ to the complex $\mathcal{C}_E^\bullet$:*

$$\boldsymbol{\Phi}(E^{[n]}) \simeq \mathcal{C}_E^\bullet.$$

The computation of the $\mathfrak{S}_n$-equivariant hypercohomology of $\mathcal{C}_E^\bullet$ would now give the cohomology $H^*(X^{[n]}, E^{[n]})$ of the tautological bundle, already computed, with different methods, by Danila [2].

4.2. The general case

For general $k > 1$ the quite complicated structure of the polygraph $D(n, k)$ does not allow any easy $\mathfrak{S}_n$-equivariant resolution of its structural sheaf as in the case $k = 1$. Nonetheless, we have a natural morphism in the equivariant derived category $\mathbf{D}_{\mathfrak{S}_n}^b(X^n)$:

$$\underbrace{\mathcal{C}_E^\bullet \otimes^L \ldots \otimes^L \mathcal{C}_E^\bullet}_{k-\text{times}} \xrightarrow{\alpha} \boldsymbol{\Phi}\big((E^{[n]})^{\otimes k}\big). \tag{7}$$

Theorem 4.2. *The mapping cone of the morphism α is acyclic in degree > 0. This means that*:

(1) $H^q(\boldsymbol{\Phi}((E^{[n]})^{\otimes k})) = 0$ *for all* $q > 0$;
(2) *in degree 0 we have an epimorphism*: $(p_* q^* E^{[n]})^{\otimes k} \to p_* q^* ((E^{[n]})^{\otimes k})$, *whose kernel is the torsion subsheaf.*

As a consequence the sheaf $\boldsymbol{\Phi}((E^{[n]})^{\otimes k}) \simeq p_* q^*((E^{[n]})^{\otimes k})$ can be identified with the term $E_\infty^{0,0}$ of the hyperderived spectral sequence:

$$E_1^{p,q} = \bigoplus_{i_1 + \cdots + i_k = p} \mathrm{Tor}_{-q}\big(\mathcal{C}_E^{i_1}, \ldots, \mathcal{C}_E^{i_k}\big) \tag{8}$$

abutting to the cohomology $H^{p+q}(\mathcal{C}_E^\bullet \otimes^L \ldots \otimes^L \mathcal{C}_E^\bullet)$ of the k-fold derived tensor product $\mathcal{C}_E^\bullet \otimes^L \ldots \otimes^L \mathcal{C}_E^\bullet$.

5. The spectral sequence of invariants and applications to cohomology for $k = 2$

Working out the term $E_\infty^{0,0}$ of the spectral sequence (8) in complete generality is hard, due to evident technical difficulties. Nonetheless, by (3), all what is really needed is the knowledge of the $\mathfrak{S}_n$-*invariants* $\boldsymbol{\Phi}((E^{[n]})^{\otimes k})^{\mathfrak{S}_n}$ of the image $\boldsymbol{\Phi}((E^{[n]})^{\otimes k})$, or equivalently, since $\pi_* \circ \boldsymbol{\Phi} = \mathbf{R}\mu_*$, the direct image $\mu_*((E^{[n]})^{\otimes k})$ of the tensor power $(E^{[n]})^{\otimes k}$ for the Hilbert–Chow morphism. The symmetric group $\mathfrak{S}_n$ acts on the derived tensor product $\mathcal{C}_E^\bullet \otimes^L \ldots \otimes^L \mathcal{C}_E^\bullet$ and

consequently on the hyperderived spectral sequence $E_1^{p,q}$. It follows, since taking $\mathfrak{S}_n$-invariants is an exact functor, that the image $\mu_*((E^{[n]})^{\otimes k})$ can be identified with the $\mathcal{E}_\infty^{0,0}$ term of the spectral sequence of invariants:

$$\mathcal{E}_1^{p,q} \simeq \left(E_1^{p,q}\right)^{\mathfrak{S}_n}. \tag{9}$$

This spectral sequence of coherent sheaves over $S^n X$ turns out to be much simpler than the original one and effectively useful, at least for $k = 2$, to compute $\mu_*((E^{[n]})^{\otimes k}) \simeq \boldsymbol{\Phi}((E^{[n]})^{\otimes k})^{\mathfrak{S}_n}$.

Theorem 5.1. *For $k = 2$ the spectral sequence of invariants $\mathcal{E}_1^{p,q}$ degenerates at level $\mathcal{E}_2$. As a consequence we obtain that, in the derived category* $\mathbf{D}^b(S^n X)$:

$$\mu_*\left(E^{[n]} \otimes E^{[n]}\right) \simeq \left(\mathcal{C}_E^\bullet \otimes \mathcal{C}_E^\bullet\right)^{\mathfrak{S}_n} \simeq 0 \longrightarrow \left(\mathcal{C}_E^0 \otimes \mathcal{C}_E^0\right)^{\mathfrak{S}_n} \xrightarrow{\mathrm{id} \otimes d_{\mathcal{C}_E^\bullet}^0} \left(\mathcal{C}_E^0 \otimes \mathcal{C}_E^1\right)^{\mathfrak{S}_n} \longrightarrow 0.$$

Theorem 1.1 is now easily obtained by applying the long exact cohomology sequence.

Acknowledgements

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