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Partial Differential Equations

# On nonlinear diffusion problems with strong degeneracy

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**Abstract**

In this Note, we study the 'triply' degenerate problem: $b(v)_t - \Delta g(v) + \operatorname{div} \Phi(v) = f$ on $Q := (0,T) \times \Omega$, $b(v(0,\cdot)) = b(v_0)$ on $\Omega$ and $g(v) = g(a)$ 'on some part of the boundary' $(0,T) \times \partial\Omega$, in the case of continuous nonhomogenous and nonstationary boundary data $a$. The functions $b, g$ are assumed to be continuous nondecreasing and to verify the normalisation condition $b(0) = g(0) = 0$ and the range condition $R(b+g) = \mathbb{R}$. Using monotonicity and penalization methods, we prove existence of a weak entropy solution in the spirit of F. Otto (1996). ***To cite this article: K. Ammar, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***


**Résumé**

**Sur les problèmes de diffusion non linéaires avec dégénérescence forte.** Dans cette Note, on étudie le problème triplement dégénéré : $b(v)_t - \Delta g(v) + \operatorname{div} \Phi(v) = f$ sur $Q := (0,T) \times \Omega$, $b(v(0,\cdot)) = b(v_0)$ dans $\Omega$ et $g(v) = g(a)$ « sur une partie de la frontière » $(0,T) \times \partial\Omega$, dans le cas d'une donnée $a$ continue non homogène et non stationnaire sur le bord. Les fonctions $b, g$ sont supposées être continues croissantes, vérifiant la condition de normalisation : $b(0) = g(0) = 0$ et de surjectivité $R(b+g) = \mathbb{R}$. En utilisant des méthodes de monotonie et de pénalisation, on prouve l'existence d'une solution entropique au sens de F. Otto (1996). ***Pour citer cet article : K. Ammar, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***


## 1. Introduction

Let $\Omega$ be strong $C^{1,1}$ bounded open subset of $\mathbb{R}^N$ with regular boundary if $N > 1$. We consider the following initial boundary value problem of parabolic-hyperbolic type:

$$P_{b,g}(v_0, a, f) \quad \begin{cases} \dfrac{\partial b(v)}{\partial t} - \Delta g(v) + \operatorname{div} \Phi(v) = f \quad \text{on } Q := (0,T) \times \Omega, \\ \text{“}g(v) = g(a) \text{ on some part of”} \ \Sigma := (0,T) \times \partial\Omega, \\ b(v)(0,\cdot) = u_0 := b(v_0) \quad \text{on } \Omega, \end{cases}$$

where $\Phi : \mathbb{R} \to \mathbb{R}^N$ is a continuous vector field, $b, g : \mathbb{R} \to \mathbb{R}$ are nondecreasing, locally Lipschitz continuous such that $b(0) = g(0) = 0$ and $R(b+g) = \mathbb{R}$. We suppose that $v_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q)$ and

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$$\begin{cases} a \in C(\Sigma) \text{ is a trace of a function } \tilde{a} \in C(Q) \text{ with } g(\tilde{a}) \in L^2(0, T, H^1(\Omega)), \\ \Delta g(\tilde{a}) \in L^1(Q) \text{ and } \tilde{a}_t \in L^1(Q). \end{cases} \tag{1}$$

Equations of this type arise in certain models of fluid flows through porous media, Stefan-type problems, and so on. In particular when $g(u) = u$, the problem is of elliptic-parabolic type and when $g(u) = 0$, $b(u) = u$, it is of hyperbolic type. In this last case, the boundary condition is understood in the sense of [2] and not in the Dirichlet sense. In a quite recent work [4], the authors have studied the problem $P_{b,g}(v_0, a, f)$ in the particular case where $b(u) = u$ and they have introduced a new formulation of the boundary conditions and proved uniqueness of a weak entropy solution and consistency with viscosity approximations. The boundary condition is given by means of a limit expressed by 'boundary layer' sequences and is a generalization of the condition proposed by F. Otto in [6]. In [5], A. Michel and J. Vovelle proposed an equivalent integral version of the so-called weak entropy condition and proved the convergence of a numerical finite volume scheme towards a generalized version called entropy-process solution. Here, we give an equivalent formulation adapted for the 'triply' degenerate case and propose a new 'analytic' proof of the existence result. Taking into account the influence of the degenerate parabolic term on the boundary conditions, we are invited to mix the techniques of [4] and [1] in order to solve the general problem. Our new formulation, clarifies in particular the relation between the formulations proposed by [3] in one hand and by [4] and [5] on the other hand.

## 2. Definitions and main results

For any $k, a \in \mathbb{R}$, for a.e. $x \in \partial\Omega$; let

$$\omega^+(x, k, a) := \max_{k \leqslant r, s \leqslant a \vee k} \left| \big(\Phi(r) - \Phi(s)\big) \cdot \eta(x) \right|,$$
$$\omega^-(x, k, a) := \max_{a \wedge k \leqslant r, s \leqslant k} \left| \big(\Phi(r) - \Phi(s)\big) \cdot \eta(x) \right|,$$

where $\eta$ denotes the unit outer normal to $\partial\Omega$. Following [5], we define an entropy solution of $P_{b,g}(v_0, a, f)$ as follows:

**Definition 2.1.** A function $v \in L^\infty(Q)$ is said to be a weak entropy solution to the problem $P_{b,g}(v_0, a, f)$ if

$$g(v) - g(a) \in L^2\big(0, T, H_0^1(\Omega)\big),$$

and $v$ satisfies the following entropy inequalities:

For all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0, T) \times \mathbb{R}^N)$ such that $\xi \geqslant 0$ and $\operatorname{sign}^+(g(a) - g(k))\xi = 0$ a.e. on $\Sigma$,

$$\begin{aligned} -\int_\Sigma \omega^+(x, k, a)\xi \leqslant \int_Q \Big\{ &\big(b(v) - b(k)\big)^+ \xi_t + \chi_{\{v>k\}}\big(\Phi(v) - \Phi(k)\big) \cdot \nabla\xi \\ &+ \chi_{\{v>k\}} f\xi - \nabla\big(g(v) - g(k)\big)^+ \cdot \nabla\xi \Big\} + \int_\Omega \big(b(v_0) - b(k)\big)^+ \xi(0, \cdot) \end{aligned} \tag{2}$$

and for all $k \in \mathbb{R}$, for all $\xi \in C_0^\infty([0, T) \times \mathbb{R}^N)$ such that $\xi \geqslant 0$ and $\operatorname{sign}^+(g(k) - g(a))\xi = 0$ a.e. on $\Sigma$,

$$\begin{aligned} -\int_\Sigma \omega^-(x, k, a)\xi \leqslant \int_Q \Big\{ &\big(b(k) - b(v)\big)^+ \xi_t + \chi_{\{k>v\}}\big(\Phi(k) - \Phi(v)\big) \cdot \nabla\xi \\ &- \chi_{\{k>v\}} f\xi - \nabla\big(g(k) - g(v)\big)^+ \cdot \nabla\xi \Big\} + \int_\Omega \big(b(k) - b(v_0)\big)^+ \xi(0, \cdot). \end{aligned} \tag{3}$$

In particular, $v$ is a weak solution of $P_{b,g}(v_0, a, f)$ and in the case where $g$ is strictly increasing, the boundary condition is satisfied in the Dirichlet sense. This definition generalizes the one introduced in [1] for the purely hyperbolic problem; and in the case where $\Phi$ is Lipschitz continuous and $b(v) = v$, it can be formulated exactly as in [5].

**Theorem 2.2.** *For any $(v_0, f) \in L^\infty(\Omega) \times L^\infty(Q)$, for any $a \in C(\Sigma)$ satisfying (1), there exists a unique function $u \in L^\infty(Q)$ such that $u = b(v)$ and $v$ is a weak entropy solution of $P_{b,g}(v_0, a, f)$.*

The uniqueness result follows as a consequence of the following $L^1$-comparison principle.

**Theorem 2.3.** *For $i = 1, 2$, let $(v_{0i}, f_i) \in L^\infty(\Omega) \times L^\infty(Q)$ and $a_i \in C(\Sigma)$ satisfying* (1) *and such that $g(a_1) \leqslant g(a_2)$ a.e. on $\Sigma$. Let $v_i \in L^\infty(Q)$ be an entropy solution of $P_{b,g}(v_{0i}, a_i, f_i)$.*

*Then there exist $\kappa \in L^\infty(Q)$ with $\kappa \in \operatorname{sign}^+(v_1 - v_2)$ a.e. in $Q$ such that, for any $\xi \in \mathcal{D}([0, T[\times \mathbb{R}^N)$, $\xi \geqslant 0$,*

$$-\int_\Sigma \omega^-(x, a_1, a_2)\xi \leqslant \int_Q \big(b(v_1) - b(v_2)\big)^+ \xi_t + \chi_{\{v_1 > v_2\}}\big(\Phi(v_1) - \Phi(v_2)\big) \cdot \nabla\xi - \int_Q \nabla\big(g(v_1) - g(v_2)\big)^+ \cdot \nabla\xi$$
$$+ \int_Q \kappa(f_1 - f_2)\xi + \int_\Omega \big(b(v_{01}) - b(v_{02})\big)^+ \xi(0, \cdot). \tag{4}$$

## 3. Proof of the existence and uniqueness result

The uniqueness is proved through the method of doubling variables of Kruzhkov and uses similar arguments as in [1]. The proof of the existence result consists of three steps: in a first step, we prove existence of a bounded entropy solution of the penalized problem with $L^\infty$ data $v_0, a, f$,

$$P_{b_l,g}(v_0, a, f, \psi) \quad \begin{cases} b_r(v)_t - \Delta g(v) + \operatorname{div}\Phi(v) + \psi(v) = f & \text{on } Q, \\ \text{“}v = a\text{” on some part of } \Sigma, \\ b_r(v(0, \cdot)) = b_r(v_0) \quad \text{in } \Omega, \end{cases}$$

where $b_r(x) = b(x) + \frac{1}{r}x$, $x \in \mathbb{R}$ and $\psi$ is an increasing Lipschitz continuous function on $\mathbb{R}$ such that $\psi(0) = 0$. This is done via approximation by ‘doubly penalized’ problems with homogeneous boundary condition of type:

$$P^{m,n}_{b_r,g}(\tilde{v}_0, 0, \tilde{f}, \psi) \quad \begin{cases} \dfrac{\partial b_r(v)}{\partial t} - \Delta g(v) + \operatorname{div}\Phi(v) + \beta_{m,n}(v) + \psi(v_{m,n}) = \tilde{f} & \text{on } \tilde{Q}, \\ g(v) = 0 & \text{on } \tilde{\Sigma}, \\ v(0, \cdot) = \tilde{v_0} & \text{on } \tilde{\Omega}. \end{cases}$$

Here, $\tilde{\Omega}$ is a Lipschitz domain strictly larger than $\Omega$ and $\tilde{Q} = (0, T) \times \tilde{\Omega}$, $\tilde{\Sigma} := (0, T) \times \partial\tilde{\Omega}$. The functions $\tilde{v}_0$ and $\tilde{f}$ being the trivial extensions by 0 of the data $v_0, f$ on the larger domain. The function $\tilde{a} \in C(Q)$ is a continuous extension onto $\tilde{Q}$ of $a$ such that $g(\tilde{a}) \in L^2(0, T, H^1_0(\tilde{\Omega}))$, $\Delta g(\tilde{a}) \in L^1(\tilde{Q})$ and $\tilde{a}_t \in L^1(\tilde{Q})$. Finally, for $m, n \in \mathbb{N}$, $\beta_{m,n}$ is the graph defined on $\mathbb{R}$ by:

$$\beta_{m,n}(t, x, r) := \chi_{\tilde{Q}\setminus Q}\big(m\big(r - \tilde{a}(x)\big)^+ - n\big(\tilde{a}(x) - r\big)\big)^+, \quad \forall r \in \mathbb{R}, \text{ a.e. } (t, x) \in \tilde{Q}.$$

Due to the Lipschitz continuity of $\beta_{m,n}$ and $\psi$, using Banach’s fixed point theorem, we prove existence of an entropy solution $v \in C([0, T]; L^1(\tilde{Q})) \cap L^\infty(\tilde{Q})$ (obtained via non-linear semi-group theory). Moreover, a comparison principle holds for entropy solutions corresponding to different penalization parameters: for any $m, m', n \in \mathbb{N}$ with $m \leqslant m'$, there exists $\kappa \in L^\infty(\tilde{Q})$ with $\kappa \in \operatorname{sign}^+(v_{m,n} - v_{m',n})$ a.e. on $\tilde{Q}$ such that, for a.e. $t \in (0, T)$,

$$\int_0^t\int_{\tilde{\Omega}} \big(\psi(v_{m',n}) - \psi(v_{m,n})\big)^+ \leqslant \int_0^t\int_{\tilde{\Omega}} \kappa\big(\tilde{f} - \beta_{m',n}(v_{m',n}) - \big(\tilde{f} - \beta_{m,n}(v_{m,n})\big)\big) \leqslant 0.$$

Consequently, $v_{m',n} \leqslant v_{m,n}$ and $v_{m,n} \leqslant v_{m,n'}$ a.e. $(t, x) \in \tilde{Q}$. This comparison result ensures the a.e. convergence of the solutions $v_{m,n}$ as, successively, $m \to \infty$ and $n \to \infty$. By a straightforward application of the maximum principle and by standard energy estimates, it can be proved that $v_{m,n}$ is bounded in $L^\infty(Q)$ and $g(v_{m,n})$ is bounded in $L^2(0, T, H^1(\Omega))$ uniformly with respect to $m, n$. This in turn implies the strong convergence of $v_{m,n}$ in $L^p(Q)$ to $v_r \in L^p(Q)$ and one can deduce that $v_r$ is a weak entropy solution of the limit problem $P_{b_r,g}(v_0, a, f, \psi)$.

In a second step, thanks to the strong perturbation term $\psi$, we prove the convergence in $L^1(Q)$ of the approximative sequence $v_r$ to $v^\psi \in L^\infty(Q)$, weak entropy solution of the limit problem $P_{b,g}(v_0, a, f, \psi)$. This allows us, in particular, to solve for $p, q \in \mathbb{N}$ the degenerate problem $P_{b,g}(v_0, a, f, p, q)$: $b(v)_t - \Delta g(v) + \operatorname{div}\Phi(v) + \frac{1}{p}v^+ - \frac{1}{q}v^- = f$ on $Q$, $b(v(0, \cdot)) = b(v_0)$ on $\Omega$ and $g(v) = g(a)$ on $\Sigma$ with $L^\infty$ data.

Finally, in the third step, using monotonicity arguments and comparison results, we prove that the sequence of entropy solutions $v_{p,q}$ associated to $P_{b,g}(v_0, a, f, p, q)$ is monotone with respect to $p$ and $q$, which ensures its a.e. convergence when $p \to +\infty$ and $q \to +\infty$. Together with the range condition, this allows to deduce compactness results in $L^1$ and the weak convergence of $g(v_{p,q})$ in $L^2(0, T, H^1(\Omega))$.

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