



Functional Analysis

Spectral pictures of 2-variable weighted shifts [☆]

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Abstract

We study the spectral pictures of (jointly) hyponormal 2-variable weighted shifts with commuting subnormal components. By contrast with all known results in the theory of subnormal single and 2-variable weighted shifts, we show that the Taylor essential spectrum can be disconnected. We do this by obtaining a simple sufficient condition that guarantees disconnectedness, based on the norms of the horizontal slices of the shift. We also show that for every $k \geq 1$ there exists a k -hyponormal 2-variable weighted shift whose horizontal and vertical slices have 1- or 2-atomic Berger measures, and whose Taylor essential spectrum is disconnected. **To cite this article:** R.E. Curto, J. Yoon, C. R. Acad. Sci. Paris, Ser. I 343 (2006).

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Résumé

Les images spectrales de shifts pondérés à 2-variables. Nous étudions les images spectrales de shifts pondérés à deux variables et (conjointement) hyponormaux possédant des composants commutants sousnormaux. À la différence de tous les résultats connus dans la théorie des shifts pondérés simples sousnormaux à deux variables, nous démontrons que le spectre essentiel de Taylor peut être déconnecté. Nous faisons cela en obtenant une condition suffisante simple qui garantit le caractère déconnecté de ce spectre, basée sur les normes des sections horizontales du shift. Nous montrons également que pour chaque $k \geq 1$ il existe un shift pondéré k -hyponormal à deux variables dont le spectre essentiel de Taylor est déconnecté. **Pour citer cet article :** R.E. Curto, J. Yoon, C. R. Acad. Sci. Paris, Ser. I 343 (2006).

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Le Problème du Relèvement des Sousnormaux Commutants (PRSC) demande quelles sont les conditions nécessaires et suffisantes pour qu'une paire commutante d'opérateurs sousnormaux dans l'espace de Hilbert \mathcal{H} admette une paire commutante d'extensions normales agissant sur un espace Hilbert $\mathcal{K} \supseteq \mathcal{H}$. Dans nos recherches récentes nous avons montré que l'hyponormalité, bien que nécessaire, n'est pas une condition suffisante pour le relèvement. Nous avons fait cela en considérant des shifts pondérés commutants en deux variables, pour lesquels nous avons développé de nouvelles techniques pour détecter leur hyponormalité et sousnormalité. À l'intérieur de la classe \mathfrak{H}_0

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de shifts pondérés de 2-variables $\mathbf{T} \equiv (T_1, T_2)$ possédant des composants sousnormaux T_1 et T_2 , nous avons obtenu dans [12] une nouvelle condition nécessaire pour l'existence d'un relèvement : les mesures de Berger des sections horizontales et verticales doivent être ordonnées de façon linéaire par rapport à la continuité absolue. Plus récemment, dans [5] nous avons donné une solution abstraite du PRSC, après avoir démontré une version multivariée du critère de Bram–Halmos [1].

Dans ce travail, nous développons des techniques et des outils nouveaux, et nous les combinons avec ceux dans [11–13,5,9] et [10], afin d'éclairer la théorie spectrale des shifts pondérés à 2-variables. Comme il est déjà bien connu, l'image spectrale d'un shift pondéré unilatéral et hyponormal W_α est facile à décrire : le spectre est un disque fermé de rayon $\|W_\alpha\|$, le spectre essentiel est un cercle de rayon $\|W_\alpha\|$, et l'indice de Fredholm est -1 dans le disque ouvert. Ainsi, d'un point de vue spectral tout shift pondéré unilatéral hyponormal de norme 1 est équivalent au shift unilatéral (non-pondéré) U_+ (qui est aussi sousnormal). Pour les shifts pondérés à 2-variables la situation est tout à fait différente, et une description complète de l'image spectrale a été donnée dans [9] et [10], en utilisant la construction du groupoïde introduite dans [16] et [15], et raffinée dans [6]. La présence de la mesure de Berger était essentielle dans l'étude du comportement asymptotique des suites de poids, et a mené à des résultats concrets sur les diverses parties du spectre de Taylor. Pour les shifts pondérés *hyponormaux* à 2-variables $\mathbf{T} \in \mathfrak{H}_0$, pourtant, l'étude des propriétés spectrales exige des techniques indépendantes, puisque aucune mesure de Berger n'est présente. Dans ce qui suit, nous présentons nombre de résultats qui mettent en relief les différences profondes entre le cas en une variable et celui en deux variables. Dans le Théorème 2.2 nous exhibons, pour la première fois, une condition suffisante qui garantit le caractère déconnecté du spectre essentiel de Taylor, notamment, $\|W_{\alpha^{(1)}}\| < \|W_{\alpha^{(0)}}\|$, où $W_{\alpha^{(j)}}$ dénote le j -ème niveau horizontal de T_1 ; dans l'Exemple 1 nous montrons que cette condition peut être présente même dans les paires hyponormales avec des sections $W_{\alpha^{(j)}}$ ($j \geq 1$) mutuellement absolument continues. Nous améliorons ce résultat en montrant dans le Théorème 2.5 qu'il est possible de former davantage de composantes connexes du spectre essentiel de Taylor, tout en préservant son caractère hyponormal, si nous utilisons des shifts pondérés de type Bergman dans chaque niveau horizontal de T_1 . Ce fait est tout-à-fait étonnant, à la lumière des résultats bien connus sur les shifts pondérés en une variable. Au cours de notre analyse, nous démontrons que pour les shifts pondérés sousnormaux en deux variables, les mesures de Berger des sections horizontales $\{W_{\alpha^{(j)}}\}_{j=1}^\infty$ sont toutes mutuellement absolument continues et, par conséquent, $\|W_{\alpha^{(j)}}\| = \|W_{\alpha^{(1)}}\|$ ($j \geq 1$). Cette nouvelle condition nécessaire est facilement calculable, et complète la condition nécessaire précédente qui se trouve dans [12, Theorem 3.3]. Ensuite, nous présentons une application à la théorie des opérateurs à une variable (Exemple 3). Finalement, le Théorème 2.7 montre que même si nous supposons que les mesures de Berger des shifts associés aux lignes horizontales et colonnes verticales sont discrètes, la k -hyponormalité conjointe [5] de \mathbf{T} (pour un $k \geq 1$ arbitraire) n'est pas une condition suffisante pour forcer l'égalité $|\sigma_{T_e}(\mathbf{T})| = \partial|\sigma_T(\mathbf{T})|$.

1. Introduction

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a commuting pair of subnormal operators on Hilbert space \mathcal{H} to admit a commuting pair of normal extensions acting on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. In recent work we have shown that (joint) hyponormality, while necessary, is not a sufficient condition for lifting [11]. We did this by appealing to commuting 2-variable weighted shifts, for which we have developed new techniques to detect their hyponormality and subnormality. Within the class \mathfrak{H}_0 of 2-variable weighted shifts $\mathbf{T} \equiv (T_1, T_2)$ with commuting subnormal components T_1 and T_2 , we obtained in [12] a new necessary condition for the existence of a lifting: the Berger measures of horizontal and vertical slices must be linearly ordered with respect to absolute continuity. More recently, we gave in [5] an abstract solution of LPCS, after proving a multivariable version of the Bram–Halmos Criterion [1].

In this Note we develop new tools and techniques, and combine them with those in [11–13,5,9] and [10], to shed light on the spectral theory of 2-variable weighted shifts. As it is well known, the spectral picture of a hyponormal unilateral weighted shift W_α is easy to describe: the spectrum is a closed disk of radius $\|W_\alpha\|$, the essential spectrum is the circle of radius $\|W_\alpha\|$, and the Fredholm index is -1 in the open disk. Thus, from a spectral perspective all norm-one hyponormal unilateral weighted shifts are equivalent to the unilateral (unweighted) shift U_+ (which is also subnormal). For subnormal 2-variable weighted shifts the situation is quite different; a complete description of the spectral picture, in the case when the intersection of the boundary of the Taylor spectrum and each coordinate plane is a circle, was given in [9] and [10]. This was done using the groupoid machinery introduced in [16] and [15], and

refined in [6]. The presence of the Berger measure was essential in the study of the asymptotic behavior of the weight sequences, and led to concrete results about the various parts of the Taylor spectrum. For *hyponormal* 2-variable weighted shifts $\mathbf{T} \in \mathfrak{H}_0$, however, the study of the spectral properties requires independent techniques, since no Berger measure is present.

In what follows, we present a number of results which highlight the deep differences between the single variable case and the 2-variable case. In Theorem 2.2 we exhibit, for the first time, a sufficient condition that guarantees the *disconnectedness* of the Taylor essential spectrum, namely, $\|W_{\alpha^{(1)}}\| < \|W_{\alpha^{(0)}}\|$, where $W_{\alpha^{(j)}}$ denotes the j -th horizontal slice of T_1 ; in Example 1 we show that this condition can even be present in hyponormal pairs with mutually absolutely continuous horizontal slices $W_{\alpha^{(j)}}$ ($j \geq 1$). We improve this by showing in Theorem 2.5 that more connected components of the Taylor essential spectrum can be formed, still preserving hyponormality, if we use Bergman-like weighted shifts on each horizontal level of T_1 . This fact is quite surprising, in view of the well known one-variable results. Along the way we prove that, for subnormal 2-variable weighted shifts, the Berger measures of the horizontal slices $\{W_{\alpha^{(j)}}\}_{j=1}^\infty$ are all mutually absolutely continuous and, as a result, $\|W_{\alpha^{(j)}}\| = \|W_{\alpha^{(1)}}\|$ (all $j \geq 1$). This new necessary condition is easily computable, and complements the previous necessary condition found in [12, Theorem 3.3]. Next, we present an application to single variable operator theory. It is well known that for T a hyponormal operator on \mathcal{H} , $r((T - \lambda)^{-1}) = \frac{1}{\text{dist}(\lambda, \sigma(T))}$ ($\lambda \notin \sigma(T)$), where r denotes spectral radius and dist denotes distance. If we substitute the left spectrum for the spectrum, the result is far from obvious; we actually show in Example 3 that for T hyponormal, the equality $\|[(T - \lambda)^{(\ell)}]^{-1}\| = \frac{1}{\text{dist}(\lambda, \sigma_\ell(T))}$ may fail, where σ_ℓ denotes the left spectrum and $[(T - \lambda)^{(\ell)}]^{-1} := [(T - \lambda)^*(T - \lambda)]^{-1/2}$ is the canonical left inverse. Finally, Theorem 2.7 shows that even if we assume that the Berger measures of the shifts associated to horizontal rows and vertical columns are discrete, joint k -hyponormality [5] of \mathbf{T} (for an arbitrary $k \geq 1$) is not sufficient to force the equality $|\sigma_{T_e}(\mathbf{T})| = \partial|\sigma_T(\mathbf{T})|$. (For compact sets $K \subseteq \mathbb{C}^2$ and $L \subseteq \mathbb{R}^2$, $|K| := \{(|z_1|, |z_2|) : (z_1, z_2) \in K\}$ and ∂L denotes the outer boundary of L , that is, the boundary of the connected component of $\mathbb{R}^2 \setminus L$.)

2. Main results

We begin by listing three basic results which are needed in the proofs of Theorems 2.2, 2.5 and 2.7, and Example 1.

Lemma 2.1. (i) ([2,3]) Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces, and let $A_i \in \mathcal{L}(\mathcal{H}_1)$, $C_i \in \mathcal{L}(\mathcal{H}_2)$ and $B_i \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, ($i = 1, \dots, n$) be such that $\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{pmatrix} := \left(\begin{pmatrix} A_1 & 0 \\ B_1 & C_1 \end{pmatrix}, \dots, \begin{pmatrix} A_n & 0 \\ B_n & C_n \end{pmatrix} \right)$ is commuting. Assume that \mathbf{A} and $\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$ are Taylor invertible. Then \mathbf{C} is Taylor invertible.

(ii) ([4,14] and [2]) For \mathbf{A} and \mathbf{B} two commuting n -tuples of bounded operators on Hilbert space, we have

$$\begin{aligned} \sigma_T(\mathbf{A} \otimes I, I \otimes \mathbf{B}) &= \sigma_T(\mathbf{A}) \times \sigma_T(\mathbf{B}), & \sigma_\ell(\mathbf{A} \otimes I, I \otimes \mathbf{B}) &= \sigma_\ell(\mathbf{A}) \times \sigma_\ell(\mathbf{B}) \quad \text{and} \\ \sigma_r(\mathbf{A} \otimes I, I \otimes \mathbf{B}) &= \sigma_r(\mathbf{A}) \times \sigma_r(\mathbf{B}), \end{aligned}$$

where σ_ℓ and σ_r denote the left and right spectra, respectively.

(iii) ([12]) Let μ be the Berger measure of a subnormal 2-variable weighted shift, and for $j \geq 0$ let ξ_j be the Berger measure of the associated j -th horizontal 1-variable weighted shift $W_{\alpha^{(j)}}$. Then $\xi_j = \mu_j^X$ (the marginal measure of μ_j), where $d\mu_j(s, t) := \frac{1}{\gamma_{0j}} t^j d\mu(s, t)$; more precisely,

$$d\xi_j(s) = \left\{ \frac{1}{\gamma_{0j}} \int_Y t^j d\Phi_s(t) \right\} d\mu^X(s),$$

where $d\mu(s, t) \equiv d\Phi_s(t) d\mu^X(s)$ is the disintegration of μ by vertical slices. A similar result holds for the Berger measure η_i of the associated i -th vertical 1-variable weighted shifts $W_{\beta^{(i)}}$ ($i \geq 0$).

Theorem 2.2. Let \mathbf{T} be the 2-variable weighted shift given by Fig. 1, let $W_{\alpha^{(j)}} := \text{shift}(\alpha_{0j}, \alpha_{1j}, \dots)$ ($j \geq 0$), $W_\beta := \text{shift}(\beta_0, \beta_1, \dots)$, and let $B := \beta_0 \text{diag}(1, \frac{\alpha_{01}}{\alpha_{00}}, \dots)$. Assume that $\|W_{\alpha^{(j)}}\| = \|W_{\alpha^{(1)}}\| < \|W_{\alpha^{(0)}}\|$ (all $j \geq 2$). Then B is a compact operator, and

$$\begin{aligned} \sigma_T(\mathbf{T}) &= (\|W_{\alpha^{(1)}}\| \cdot \bar{\mathbb{D}} \times \|W_\beta\| \cdot \bar{\mathbb{D}}) \cup (\|W_{\alpha^{(0)}}\| \cdot \bar{\mathbb{D}} \times \{0\}), \\ \sigma_{T_e}(\mathbf{T}) &= (\|W_{\alpha^{(1)}}\| \cdot \bar{\mathbb{D}} \times \|W_\beta\| \cdot \mathbb{T}) \cup (\|W_{\alpha^{(1)}}\| \cdot \mathbb{T} \times \|W_\beta\| \cdot \bar{\mathbb{D}}) \cup (\|W_{\alpha^{(0)}}\| \cdot \mathbb{T} \times \{0\}). \end{aligned}$$

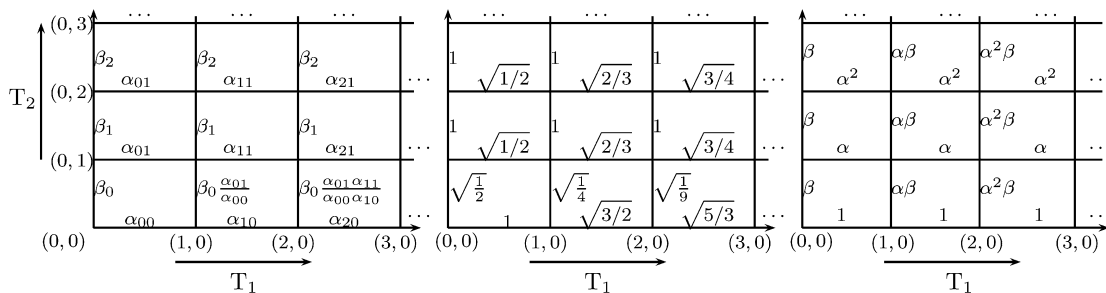


Fig. 1. Weight diagrams of the 2-variable weighted shifts in Theorem 2.2 and Examples 1 and 2, respectively.

Fig. 1. Diagrammes de poids des shifts pondérés en deux variables dans le Théorème 2.2 et les Exemples 1 et 2, respectivement.

In particular, $\sigma_{T_e}(\mathbf{T}) \neq \partial\sigma_T(\mathbf{T})$. (Here $\bar{\mathbb{D}}$ denotes the closure of the open unit disk \mathbb{D} , \mathbb{T} the unit circle, and $\partial K := \{(z_1, z_2) \in \mathbb{C}^2 : (|z_1|, |z_2|) \in \partial|K|\}$).

Theorem 2.3. For $i, j \geq 1$, let $W_{\alpha^{(j)}}$ (resp. $W_{\beta^{(i)}} := \text{shift}(\beta_{i0}, \beta_{i1}, \dots)$) be the j -th horizontal slice (resp. i -th vertical slice) of a subnormal 2-variable weighted shift. Then $\|W_{\alpha^{(j)}}\| = \|W_{\alpha^{(1)}}\|$ and $\|W_{\beta^{(i)}}\| = \|W_{\beta^{(1)}}\|$.

Proof. As a consequence of Lemma 2.1(iii), we see that $\xi_j \approx \xi_1$ and $\eta_i \approx \eta_1$ for all $i, j \geq 1$. It follows that $\text{supp } \xi_j = \text{supp } \xi_1$ and $\text{supp } \eta_i = \text{supp } \eta_1$ for all $i, j \geq 1$. Since the norm of a subnormal unilateral weighted shift always equals the supremum of the support of its Berger measure [1], the result follows. \square

Example 1 below provides a concrete instance of Theorem 2.2, with all horizontal rows admitting continuous Berger measures. We first need some notation and a few definitions. Let μ be a Reinhardt measure on \mathbb{C}^n . The set of bounded point evaluations for μ is $b.p.e(\mu) = \{\lambda \in \mathbb{C}^n : p \rightarrow p(\lambda), p \in \mathbb{C}[z], \text{ extends to a bounded point evaluation from } P^2(\mu) \text{ to } \mathbb{C}\}$. The kernel function associated with μ is

$$k(z, w) \equiv k^{(\mu)}(z, w) := \sum_{\alpha \in \mathbb{Z}_+^n} \frac{z^\alpha \bar{w}^\alpha}{\|z^\alpha\|^2},$$

and the set of convergence of k is $\mathcal{C}(k) := \{\lambda \in \mathbb{C}^n : k(\lambda, \lambda) < \infty\}$.

Example 1. Let $\mathbf{T} \equiv (T_1, T_2)$ be the 2-variable weighted shift whose weight diagram is given in Fig. 1. Then $\mathbf{T} \in \mathfrak{H}_0$, \mathbf{T} is hyponormal, $\xi_j \approx \xi_1$ ($j \geq 1$), $\eta_i \approx \eta_1$ ($i \geq 0$) and $\text{supp } \eta_i = \{0, 1\}$ ($i \geq 0$). However, \mathbf{T} is not subnormal. Moreover $\sigma_T(\mathbf{T}) = \sigma_r(\mathbf{T}) = \overline{\mathcal{C}(k)} = (\bar{\mathbb{D}} \times \bar{\mathbb{D}}) \cup (\sqrt{2} \cdot \bar{\mathbb{D}} \times \{0\})$, $\sigma_{T_e}(\mathbf{T}) = \sigma_{r_e}(\mathbf{T}) = (\sqrt{2}\mathbb{T}, 0) \cup ((\mathbb{T} \times \bar{\mathbb{D}}) \cup (\bar{\mathbb{D}} \times \mathbb{T}))$, and $\partial\sigma_T(\mathbf{T}) \neq \sigma_{T_e}(\mathbf{T})$ (cf. Fig. 2).

Theorem 2.5 below shows that, for every $k \geq 1$, it is possible to create k connected components in the σ_{T_e} of a 2-variable weighted shift \mathbf{T} , while maintaining the hyponormality of $\mathbf{T} \in \mathfrak{H}_0$. We recall that for $\ell \geq 1$, the Bergman-like weighted shift on $\ell^2(\mathbb{Z}_+)$ is $B_+^{(\ell)} := \text{shift}(\{\sqrt{\ell - \frac{1}{n+2}} : n \geq 0\})$; in particular, $B_+^{(1)} \equiv B_+$ is the Bergman shift. Note that $\|B_+^{(\ell)}\| = \sqrt{\ell} \geq 1$, $B_+^{(\ell)}$ is subnormal, with Berger measure ξ_ℓ ($\ell \geq 1$), and $d\xi_1(s) = ds$ on $[0, 1]$ and $d\xi_2(s) = \frac{s ds}{\pi\sqrt{2s-s^2}}$ on $[0, 2]$ (cf. [7,13]). The following result is a variation of [13, Theorem 3.14].

Theorem 2.4. For every $k \geq 1$ there exists (i) a family of $\{B_+^{(\ell_j)}\}_{j=0}^{k-1}$ of Bergman-like weighted shifts, and (ii) a subnormal weighted shift $W_\beta \equiv \text{shift}(\beta_0, \beta_1, \dots)$ (with $\beta_n < \beta_{n+1}$ all $n \geq 0$), such that the commuting 2-variable weighted shift \mathbf{T} with a weight diagram whose first k rows are $B_+^{(\ell_0)}, \dots, B_+^{(\ell_{k-1})}$, whose remaining rows are all equal to U_+ , and whose 0-th column is given by W_β , is hyponormal.

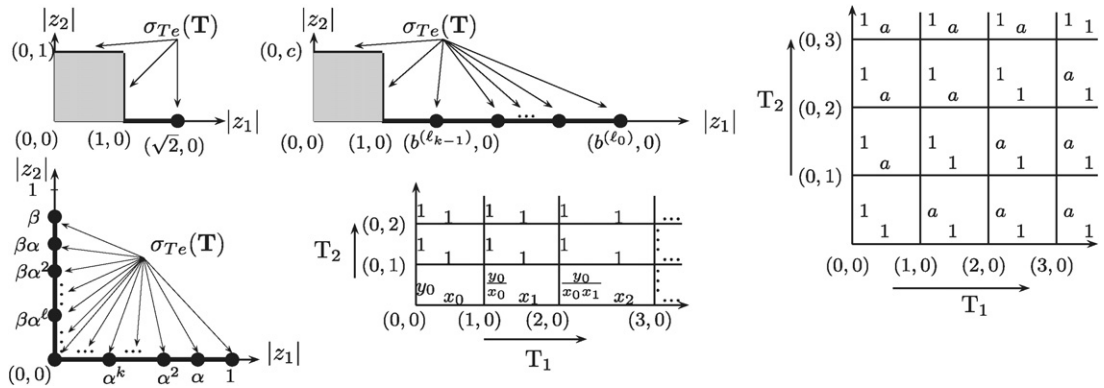


Fig. 2. (Top left and center): Spectral pictures in Example 1 and Theorem 2.5, respectively; (bottom left) spectral picture in Example 2; (top right and bottom center) weight diagrams of the 2-variable weighted shifts in Example 3 and Theorem 2.7, respectively.

Fig. 2. (En haut à gauche et au centre) : Les images spectrales dans l'Exemple 1 et le Théorème 2.5, respectivement ; (en bas à gauche) l'image spectrale dans l'Exemple 2 ; (en haut à droite et en bas au centre) des diagrammes de poids des shifts pondérés en deux variables dans l'Exemple 3 et le Théorème 2.7, respectivement.

Theorem 2.5. Let \mathbf{T} be a hyponormal 2-variable weighted shift satisfying the hypotheses in Theorem 2.4, and let $b^{(\ell_j)} := \|B_+^{(\ell_j)}\|$ ($j = 0, \dots, k - 1$) and $c := \|W_\beta\|$. Then

$$\sigma_{\mathbf{T}}(\mathbf{T}) = (\overline{\mathbb{D}} \times c\overline{\mathbb{D}}) \cup (b^{(\ell_0)}\overline{\mathbb{D}} \times \{0\}), \quad \sigma_{T_e}(\mathbf{T}) = [\overline{\mathbb{D}} \times \{c\}] \cup [\{1\} \times c\overline{\mathbb{D}}] \cup [(b^{(\ell_0)}\mathbb{T} \cup \dots \cup b^{(\ell_{k-1})}\mathbb{T}) \times \{0\}]$$

(cf. Fig. 2).

Example 2 shows that, if we do not insist that $\mathbf{T} \in \mathfrak{H}_0$ be hyponormal, the Taylor essential spectrum can consist of infinitely many circles converging to a single point. We see in particular that two commuting subnormals do not necessarily give rise to a hyponormal pair.

Example 2. Let \mathbf{T} be the 2-variable weighted shift whose weight diagram is given by Fig. 1, where $\alpha < \beta \leq 1$. Then $\mathbf{T} \in \mathfrak{H}_0$, with 1-atomic Berger measures for all horizontal and vertical slices. However, \mathbf{T} is not hyponormal. Moreover, $\sigma_{\mathbf{T}}(\mathbf{T})$ has empty interior; in fact,

$$\sigma_{\mathbf{T}}(\mathbf{T}) = \sigma_r(\mathbf{T}) = [(|z_1| \leq 1) \times \{0\}] \cup [\{0\} \times (|z_2| \leq \beta)], \quad \sigma_\ell(\mathbf{T}) = \sigma_{\ell_e}(\mathbf{T}) = \sigma_{r_e}(\mathbf{T}),$$

$$\sigma_{T_e}(\mathbf{T}) = \{(0, 0)\} \cup \left\{ \left[\bigcup_{k=0}^{\infty} (|z_1| = \alpha^k) \right] \times \{0\} \right\} \cup \left\{ \{0\} \times \left[\bigcup_{\ell=0}^{\infty} (|z_2| = \beta\alpha^\ell) \right] \right\} \quad \text{and} \quad \sigma_\ell(\mathbf{T}) = \sigma_{T_e}(\mathbf{T})$$

(cf. Fig. 2).

We now turn to a simple application of 2-variable weighted shifts to a single variable problem. Example 3 shows that there exists a hyponormal $T \in \mathcal{L}(\mathcal{H})$ with $\|(T - \lambda)^{(\ell)-1}\| \neq \frac{1}{\text{dist}(\lambda, \sigma_\ell(T))}$ for some $\lambda \notin \sigma_\ell(T)$.

Example 3. Let \mathbf{T} be the 2-variable weighted shift whose weight diagram is given by Fig. 2, where $a < 1$. Then (i) \mathbf{T} is not hyponormal, and $\sigma_\ell(\mathbf{T}) = \sigma_{\ell_e}(\mathbf{T}) = (a\mathbb{T} \times \mathbb{T}) \cup (\mathbb{T} \times a\mathbb{T})$; (ii) T_1 is hyponormal, $\sigma_\ell(T_1) = \sigma_{\ell_e}(T_1) = a\mathbb{T} \cup \mathbb{T}$ and at least one horizontal slice $W_{\alpha^{(j)}}$ of T_1 does not satisfy the identity $\|(W_{\alpha^{(j)}} - \lambda)^{(\ell)-1}\| = \frac{1}{\text{dist}(\lambda, \sigma_\ell(W_{\alpha^{(j)}}))}$ for all $\lambda \in \sigma_\ell(W_{\alpha^{(j)}})$.

The next result, Theorem 2.7, shows that k -hyponormality ($k \geq 1$) is not sufficient to guarantee $\partial\sigma_{\mathbf{T}}(\mathbf{T}) = \sigma_{T_e}(\mathbf{T})$. We first recall that if \mathbf{T} is a 2-variable weighted shift with weight sequences $\alpha \equiv \{\alpha_{\mathbf{m}}\}$ and $\beta \equiv \{\beta_{\mathbf{m}}\}$, then \mathbf{T} is k -hyponormal $\Leftrightarrow M_{\mathbf{m}}(k) \geq 0$ for all $\mathbf{m} \in \mathbb{Z}_+^2$, where $M_{\mathbf{m}}(k)$ is the $(k + 1) \times (k + 1)$ matrix of moments beginning at \mathbf{m} [5, Theorem 3.1].

Lemma 2.6. (i) For $\kappa > 1$, let $x \equiv \{x_n\}_{n=0}^\infty$ where $x_0 := \sqrt{\frac{1+\kappa}{2}}$ and $x_n := \sqrt{\frac{1+\kappa^{n+1}}{1+\kappa^n}}$ ($n \geq 1$). Then $W_\kappa := \text{shift}(x_0, x_1, \dots)$ is subnormal, with Berger measure $\xi_\kappa = \frac{1}{2}(\delta_1 + \delta_\kappa)$.

(ii) Given $k \geq 1$ and $m_1 \geq 0$, consider

$$M(m_1) := \begin{pmatrix} \gamma_{m_1}(W_\kappa) & \cdots & \gamma_{m_1+k}(W_\kappa) \\ \vdots & \ddots & \vdots \\ \gamma_{m_1+k}(W_\kappa) & \cdots & \gamma_{m_1+2k}(W_\kappa) \end{pmatrix} \quad \text{and} \quad \mathbf{1} := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

If $0 < y_0^2 \leq \frac{1}{2}$ then $M(m_1) - y_0^2 \cdot \mathbf{1} \geq 0$, where $\mathbf{1}$ is a $(k+1) \times (k+1)$ matrix.

Theorem 2.7. Let \mathbf{T} be the 2-variable weighted shift whose weight diagram is given in Fig. 2. Given $k \geq 1$, $\kappa > 1$ and W_κ , there exists $0 < y_0 \leq \sqrt{\frac{1}{2}}$ which makes \mathbf{T} k -hyponormal. Moreover, $\sigma_T(\mathbf{T}) = \sigma_r(\mathbf{T}) = (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \cup (\sqrt{\kappa} \cdot \overline{\mathbb{D}} \times \{0\})$, $\sigma_{T_e}(\mathbf{T}) = \sigma_{r_e}(\mathbf{T}) = (\sqrt{\kappa} \mathbb{T} \times 0) \cup ((\mathbb{T} \times \overline{\mathbb{D}}) \cup (\overline{\mathbb{D}} \times \mathbb{T}))$, and $\partial \sigma_T(\mathbf{T}) \neq \sigma_{T_e}(\mathbf{T})$.

Sketch of proof. Observe that the restriction of \mathbf{T} to $\vee\{e_{(m_1, m_2)} : m_2 \geq 1\}$ is unitarily equivalent to $(I \otimes U_+, U_+ \otimes I)$. We then apply [5, Theorem 3.1] to $\mathbf{m} := (m_1, 0)$, with $m_1 \geq 0$. We have $M_{(m_1, 0)}(k) \geq 0$ (all $k \geq 1$) $\Leftrightarrow M(m_1) - y_0^2 \cdot \mathbf{1} \geq 0$. We then use the moments associated with \mathbf{T} , matrix row operations, Choleski's algorithm, and a choice of $y_0 \leq \sqrt{\frac{1}{2}}$, together with Lemma 2.6(i), to show that \mathbf{T} is k -hyponormal. To calculate $\sigma_T(\mathbf{T})$ and $\sigma_{T_e}(\mathbf{T})$, we use Lemma 2.1, the projection property for the Taylor spectrum, and direct verification of the exactness of the Koszul complex associated with \mathbf{T} at the middle stage, using the technique in [8, Corollary 4.3(ii)]. \square

Remark 1. Careful analysis of the proof of Theorem 2.7 reveals that for $k \geq 1$ and $y_0 \leq \sqrt{\frac{1}{2}}$, \mathbf{T} is k -hyponormal. Thus, \mathbf{T} is indeed subnormal in the interval $(0, \sqrt{\frac{1}{2}}]$, and its Berger measure is $\mu = y_0^2 \delta_1 \times \delta_1 + (\xi_\kappa - y_0^2 \delta_1) \times \delta_0$.

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