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Geometry

On the length of simple closed quasigeodesics on convex surfaces

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Abstract

We establish, for general convex surfaces, inequalities involving the diameter, the area and the lengths of simple closed (quasi)geodesics. ***To cite this article: J. Itoh, C. Vîlcu, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Sur la longueur des quasigéodésiques simples fermées sur des surfaces convexes. On etablit, pour des surfaces convexes arbitraires, des inégalités impliquant le diamètre, l'aire et les longueurs des (quasi)géodésiques simples fermées. ***Pour citer cet article : J. Itoh, C. Vîlcu, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Une *surface convexe* S est la frontière d'un corps convexe dans l'espace d'Euclide à 3 dimensions, ou le double d'un corps convexe planaire. La métrique de S est définie, pour tout couple de points x, y sur S, comme la longueur d'un *segment* (i.e., un plus court chemin sur S) de x à y. On appelle *géodésique* une courbe de S qui est localement un segment.

Ainsi, toute surface convexe peut ne pas avoir de géodésiques fermées (voir [1] pp. 377–378 ou [5]), mais il y existe toujours au moins trois quasigéodésiques simples fermées, le cas dégénéré y compris [10]. Voir la version anglaise, ou [3] p. 114, pour la définition d'une quasigéodésique. Pour cette raison, nous avons choisi de prendre en considération les quasigéodésiques au lieu des géodésiques, mais le lecteur pourra simplement considérer des surfaces $\mathcal{C}^2$-différentiables, cas dans lequel toute quasigéodésique est nécessairement une géodésique ([3] pp. 114 et 27). Un cas d'égalité dans le Théorème 0.1, ainsi que la remarque qui le suit, offrent d'autres raisons pour notre choix.

Soit A l'aire de S, D son diamètre, et l_0 la longueur de la plus courte quasigéodésique fermée non-triviale de S (qui est simple, par des approximations lisses et d'après le Théorème D de [4]).

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Théorème 0.1. *Soit O une quasigéodésique simple fermée de longueur l sur une surface convexe S, S_1 un sous-ensemble connexe de S dont O est la frontière, d_1 la distance maximale de O aux points de S_1, et A_1 l'aire de S_1. Alors*

$$A_1 \leqslant l d_1 \leqslant 2A_1.$$

La première égalité est valable si et seulement si S_1 est le double d'un rectangle coupé le long d'un côté. La deuxième égalité est valable si et seulement si S_1 est isométrique à la limite d'une suite de parties latérales de pyramides dont toutes les faces ont la même hauteur.

L'énoncé du Théorème 0.1 peut être formulé d'une manière équivalente en considérant la frontière O d'un *domaine convexe* (i.e., un ouvert homéomorphe au disque, contenant avec tout couple de points un segment les joignant), de courbure non-négative (à la Alexandrov [1]). Voir [13] pour d'autres propriétés de tels domaines, munis avec des métriques analytiques.

Le *spectre des longueurs* de S est l'ensemble de toutes les longueurs des quasigéodésiques simples fermées de S. On peut montrer, en appliquant le Théorème 0.1, qu'*une surface convexe a le spectre des longueurs non-borné si et seulement si elle est un tétraèdre isoscèle* [7].

Corollaire 0.2. *Soit l_E la longueur de l'équateur de la surface convexe de rotation S. On a $A < l_E D \leqslant 2A$, et l'égalité est valable si et seulement si S est l'union de deux cônes.*

Tandis que la première inégalité ci-dessus est valable pour toute surface convexe en considérant l_0 au lieu de l_E (voir le Corollaire 0.5), nous ne le savons pas pour la deuxième égalité, et nous présentons dans la suite quelques variantes plus faibles.

Corollaire 0.3. *Pour toute surface convexe S et toute quasigéodésique simple fermée de S de longueur l, on a $l^2 - 2Dl + 4A > 0$.*

L'énoncé suivant met en évidence une lacune dans le spectre des longueurs des surfaces convexes « longues », autour de la valeur du D.

Corollaire 0.4. *Si $D^2 \geqslant 4A$ alors il n'existe aucune quasigéodésique simple fermée de S de longueur l dans l'intervalle* $[D - \sqrt{D^2 - 4A}, D + \sqrt{D^2 - 4A}]$.

On peut utiliser le Corollaire 0.3 et des bornes sur l_0 ou sur l, pour obtenir d'autres inéqualités. Les Corollaires 0.5 et 0.7 en constituent deux exemples.

Corollaire 0.5. *Pour toute surface convexe S on a $A < l_0 D < 34A$.*

La conjecture de A.D. Alexandrov, disant que $A \leqslant \frac{\pi}{2} D^2$ (voir [1] p. 417, ou [2] p. 42), n'est pas résolue. La meilleure borne connue, $A \leqslant \frac{8}{\pi} D^2 \approx 2,546 D^2$, est due à E. Calabi et J. Cao [4].

Corollaire 0.6. *Étant donnée une constante C telle que $l_0 \leqslant CD$, on a $A < CD^2$.*

A. Nabutovsky et R. Rotman [9], et S. Sabourau [12], ont démontrée que $l_0 \leqslant 4D$. R. Rotman [11] a amélioré substantiellement cette évaluation pour de grandes classes de surfaces, pour lesquelles on a des meilleures bornes dans le Corollaire 0.6, mais la question de savoir si généralement $l_0 \leqslant 2D$ reste ouverte.

V.A. Toponogov a établi une borne supérieure pour la longueur d'une géodésique fermée d'une surface convexe lisse, relativement à une borne inférieure sur la courbure (voir [14] ou [8] p. 297). Cela implique :

Corollaire 0.7. *Pour toute surface convexe S $\mathcal{C}^2$-différentiable, dont la courbure de Gauss K vérifie $K \geqslant k > 0$, et pour toute géodésique simple fermée de longueur l sur S, on a $lD < 2A + \frac{2\pi^2}{k}$.*

1. Introduction

A *convex surface* S is the boundary of a *convex body* (compact convex set with interior points) in the Euclidean space $\mathbb{R}^3$, or a doubly covered planar convex body. Its metric ρ is defined, for any points x, y in S, as the length $\rho(x, y)$ of a *segment* (i.e., shortest path on S) joining x to y. A *geodesic* is a curve which is locally a segment.

As such, a convex surface may lack many of the properties usual for the Riemannian geometer. For example, it may have no closed geodesic, as pointed out for polyhedral surfaces by A.D. Alexandrov (see [1] pp. 377–378), and proved for typical (in the sense of Baire category) surfaces by P. Gruber [5]. Nevertheless, there always exist at least three simple closed quasigeodesics (the degenerate case included), by a result of A.V. Pogorelov [10]. While any geodesic of a convex surface S is a quasigeodesic, the converse is true if S has bounded specific curvature, in particular if S is $\mathcal{C}^2$-differentiable (see [3] pp. 114 and 27). This is why we will refer more generally to quasigeodesics instead of geodesics, but the reader may simply consider only $\mathcal{C}^2$-surfaces. An equality case in Theorem 2.1, as well the remark following it, offer other reasons for our choice.

Denote by A the area of a convex surface S, by D the diameter of S, $D = \max_{x,y\in S} \rho(x, y)$, and by l_0 the length of a shortest non-trivial closed quasigeodesic on S (which is simple, by smooth approximation and a result of E. Calabi and J. Cao [4]). Our main result has several consequences, involving these quantities and also concerning the *length spectrum* of S (i.e., the set of all lengths of simple closed quasigeodesics on S).

For the reader's convenience, we recall next the definition of a quasigeodesic. Consider a broken geodesic Γ which is a Jordan arc, say $\Gamma = \bigcup_{i=0}^{n} \Gamma_{a_i a_{i+1}}$, where $\Gamma_{a_i a_{i+1}}$ is a geodesic arc joining the points $a_i, a_{i+1} \in S$ $(i = 0, \ldots, n)$. Then a *right* and a *left side* can be consistently locally defined along $\Gamma \setminus \{a_0, a_{n+1}\}$. Denote by α_i and β_i the angle between $\Gamma_{a_i a_{i-1}}$ and $\Gamma_{a_i a_{i+1}}$ to the right and to the left of Γ, respectively. The *right* and *left swerve* of Γ are the numbers $s_r(\Gamma) = \sum_{i=1}^{n}(\pi - \alpha_i)$, $s_l(\Gamma) = \sum_{i=1}^{n}(\pi - \beta_i)$. Consider now a Jordan arc A which has definite directions at its endpoints p, q, and Γ a broken geodesic from p to q which is a Jordan arc and lies to the right of, or on, A. Denote by δ_p and δ_q the angles between Γ and A at p and q. Then $\lim(\delta_p + \delta_q + s_r(\Gamma))$ exists when Γ approaches A from the right (see [1] p. 353) and it is called the *right swerve* of A ([3] p. 109). The *left swerve* is defined similarly. A *quasigeodesic arc* is a Jordan arc which has definite directions at each point and every subarc of which has nonnegative right and left swerves ([3] p. 114).

A segment connecting two points with complete angles $\leqslant \pi$ forms, traversed back and forth, a *degenerate closed quasigeodesic* ([3] p. 114).

Our main tools will be cut loci. A *segment between a point x and a closed set K not containing x* is a segment from x to a point in K, not longer than any other such segment. The *cut locus $C(K)$* of the closed set $K \subset S$ is the set of all points $y \in S$ such that there is a segment from y to K not extendable as a segment beyond y.

2. Estimates

We define next a class Λ of surfaces, to be used within Theorem 2.1. Let Π be a plane in $\mathbb{R}^3$, C a circle in Π of centre o, and K a finite or countable subset of C. Construct, for any $x \in K$, the line $T_x \subset \Pi$ tangent to C at x. For any point y on the line orthogonal to Π at o, and any closed convex curve $O \subset C \cup \bigcup_{x\in K} T_x$, set $P_{y,O} = \mathrm{cl}((\mathrm{bd\,conv}(\{y\} \cup O)) \setminus \Pi)$. Define $\Lambda = \bigcup_{\Pi,C,K,O,y} P_{y,O}$.

Denote by $\mathcal{L}_h$ the family of the lateral parts of the pyramids all of whose faces have the same height h, and notice that any surface in Λ is the limit (with respect to the usual Pompeiu–Hausdorff metric) of a sequence of surfaces in some $\mathcal{L}_h$, and any such limit is in Λ, hence $\Lambda = \bigcup_h \mathrm{cl}\mathcal{L}_h$.

Theorem 2.1. *If O is a simple closed quasigeodesic of length l on a convex surface S, S_1 a connected subset of S bounded by O, d_1 the maximal distance from O to points in S_1, and A_1 the area of S_1, then*

$$A_1 \leqslant l d_1 \leqslant 2A_1.$$

The first equality holds if and only if S_1 is a doubly covered rectangle cut along one edge, and the second equality holds if and only if S_1 is isometric to a surface in Λ.

Proof. Approximate S by a polyhedral convex surface P such that O is approximated by a simple closed quasigeodesic on P. Since the inequalities we want to prove remain true by taking the limit ([3] p. 107), we may assume that S is polyhedral.

The idea for the first part is borrowed from [2] p. 42. Simply cut along the cut locus $C(O)$ of O in S_1, which is a tree containing all vertices of S_1. The result E is (isometric to) a set included either in a cylinder Cy if O is a geodesic, or in a cone Co over O with points in O where the angle towards Co is $< \pi$ if O is a quasigeodesic but not a geodesic. In both cases we get basis l and height d_1, and the first inequality follows easily. If equality holds then we are in the first case and the closure of E in Cy equals Cy. Moreover, since S is convex, $C(O)$ has to be an arc, all points of which are at maximal distance to O.

For the second part, choose a point $x \in S_1$ with $\rho(x, O) = d_1$. Then, from the first variation formula and the definition of a quasigeodesic, for every $x_0 \in X$ with $\rho(x, x_0) = d_1$, there exists a unique segment $\gamma_{x_0 x}$ from x_0 to x and $\gamma_{x_0 x} \perp O$ at x_0.

Glue two isometric copies of S_1 along the boundary O. By Alexandrov's gluing theorem ([1] p. 362 or [3] p. 154) and Pogorelov's rigidity theorem ([10] p. 167), the result R is isometric to a unique (up to rigid motions) convex surface. Because segments of convex surfaces do not branch and R is symmetric with respect to O, S_1 is strongly convex in R, i.e. it contains, with any two points, all segments joining the points. While distances in S_1 with respect to the metric of R may increase, compared to those in S, the distance from x to O remains the same.

Next we refer to R. Let $C(x)$ denote the cut locus of x (it is a tree), and $C = C(x) \cap O$. For every point $y \in C$, let D_y denote the maximal (with respect to inclusion) geodesic digon included in S_1, bounded by (possibly coinciding) segments from y to x. By Gauss–Bonnet theorem, each $D_y \neq \emptyset$ contains at least one vertex of R, and all vertices of $\operatorname{int} R \cap S_1$ are included in $\operatorname{int} \bigcup_{y \in C} D_y$. Moreover, since segments do not bifurcate and do not pass beyond conical points, we have $O \subset \operatorname{cl}(S_1 \setminus \bigcup_{y \in C} D_y)$.

Cut off all D_y's from S_1. The result L, after identifying the boundary segments of each D_y, is isometric to the lateral part of a pyramid of vertex x over O, the base vertices corresponding to the points of O where the angle towards L is $< \pi$, which are precisely $y \in C$.

Since O is a simple closed quasigeodesic of S, and consequently of R, for any point $y \in O$ the angle of the semi-tangents τ_y^+, τ_y^- to O at y is at most π. Then, if one of the (planar) triangles of $\operatorname{cl}(S_1 \setminus \bigcup_{y \in C} D_y)$ is obtuse at its vertex $y \in C$, then its neighbour is acute at y, and thus the height of the former is larger than the height of the later. Therefore, we still have on L $\rho^L(x, O) = d_1$, and the conclusion follows from $l d_1/2 \leqslant A_L \leqslant A_1$.

For the second equality, necessarily the polyhedral approximation (of) S_1 coincides to $L \in \mathcal{L}_{d_1}$, and the rest follows from $\Lambda = \bigcup_h \operatorname{cl} \mathcal{L}_h$. □

The statement of Theorem 2.1 can equivalently be given for the boundary O of a *convex domain* (i.e., open set homeomorphic to the disk, containing with any two points a segment joining the points), nonnegatively curved (in the sense of A.D. Alexandrov [1]). For, observe that the result R of gluing together two copies of such a domain along the boundary O is (isometric to) a convex surface, and O is a simple closed quasigeodesic (but not necessarily a geodesic) of R (see [3] p. 154). Refer to [13] for other properties of such domains, endowed with real analytic metrics.

We mention here an application of Theorem 2.1, in a direction different to the topic of this Note, namely that *a convex surface has unbounded length spectrum if and only if it is an isosceles tetrahedron* [7].

The *equator* (i.e., the largest parallel circle) of any convex surface of revolution is easily seen to be a closed quasigeodesic. Since the diameter of such a surface is realized between its poles, we have:

Corollary 2.2. *If l_E denotes the length of the equator of the convex surface of revolution S then $A < l_E D \leqslant 2A$, and the equality holds if and only S is the union of two cones.*

While, by considering above l_0 instead of l_E, the first inequality holds for any convex surface (see Corollary 2.5), we do not have a proof for the second inequality in the general case, and present in the following some weaker forms. Since the length spectrum of isosceles tetrahedra is unbounded, $lD \leqslant 2A$ is not generally true.

Corollary 2.3. *For any convex surface S and any simple closed quasigeodesic of length on l on S holds $l^2 - 2Dl + 4A > 0$.*

Proof. Let O be a simple closed quasigeodesic on S of length l, x, y points in S realizing the diameter of S, and x_0, y_0 projections of x, y on O. Define $S_2 = \operatorname{cl}(S \setminus S_1)$ and d_2, A_2 in the obvious way.

Assume first that x and y belong to different components of $S \setminus O$. Then $D = \rho(x, y) \leqslant \rho(x, x_0) + \rho(x_0, y_0) + \rho(y_0, y) \leqslant d_1 + l/2 + d_2$. Observe that at least one of the above inequalities is strict, because the segments joining x and y to their projections on O are orthogonal to O.

Assume now that x and y belong to the same component of $S \setminus O$, say S_1, and $\rho(x, x_0) \geqslant \rho(y, y_0)$. Let γ_{xx_0} be the segment from x to x_0, and $x_1 \in \gamma_{xx_0}$ such that $\rho(x_1, x_0) = \rho(y, y_0)$. It follows, from Theorem 1 in [6], that $\rho(x_1, y) \leqslant \rho(x_0, y_0) \leqslant l/2$, hence $D = \rho(x, y) < \rho(x, x_1) + \rho(x_1, y) \leqslant d_1 + l/2 < d_1 + l/2 + d_2$.

The last relation and $l(d_1 + d_2) \leqslant 2A$ obtained from Theorem 2.1 simply write as $2A \geqslant l(d_1 + d_2) \geqslant l(D - l/2)$ and the conclusion follows. □

The next statement highlights a gap in the length spectrum of 'long' convex surfaces, around the value of D. It follows directly from the second order inequality in l obtained at Corollary 2.3.

Corollary 2.4. *If $D^2 \geqslant 4A$ then there exists no simple closed quasigeodesic of S with length in* $[D - \sqrt{D^2 - 4A}, D + \sqrt{D^2 - 4A}]$.

One can use Corollary 2.3 and bounds on l_0 or on l, in order to obtain other inequalities. Our Corollaries 2.5 and 2.7 constitute two such examples.

Corollary 2.5. *For any convex surface S holds $A < l_0 D < 34A$.*

Proof. With the notations of Theorem 2.1 and the proof of Corollary 2.3, choose $x \in S_1$, $y \in S_2$ and $x_0, y_0 \in O$ such that $\rho(x, x_0) = d_1$ and $\rho(y, y_0) = d_2$. Also choose a segment γ_{xy} from x to y and put $\{z\} = \gamma_{xy} \cap O$. Then $d_1 + d_2 = \rho(x, x_0) + \rho(y, y_0) \leqslant \rho(x, z) + \rho(z, y) = \rho(x, y) \leqslant D$, so the first inequality in Theorem 2.1 becomes $A \leqslant l_0(d_1 + d_2) \leqslant l_0 D$. Finally observe that one cannot have both $A = l_0(d_1 + d_2)$ and $d_1 + d_2 = D$, so the inequality is strict.

The second part follows from Corollary 2.3 and the estimate $l_0 \leqslant 8\sqrt{A}$ obtained in [9] and [12]. □

An old conjecture of A.D. Alexandrov states that $A \leqslant \frac{\pi}{2} D^2$ (see [1] p. 417, or [2] p. 42). The best achieved bound, $A \leqslant \frac{8}{\pi} D^2 \approx 2.546 D^2$, is due to E. Calabi and J. Cao [4]. By the use of Corollary 2.5 we obtain the following.

Corollary 2.6. *Given a constant C such that $l_0 \leqslant CD$, we have $A < CD^2$.*

A. Nabutovsky and R. Rotman [9], and independently S. Sabourau [12], proved that $l_0 \leqslant 4D$. R. Rotman [11] considerably improved this estimate for large classes of surfaces, for which it yields better estimates in Corollary 2.6, but it is an open question whether $l_0 \leqslant 2D$ in general.

V.A. Toponogov settled the upper bound $\frac{2\pi}{\sqrt{k}}$, for the length of a simple closed geodesic on a smooth convex surface, in terms of a lower bound k on the curvature (see [14] or [8] p. 297). This and Corollary 2.3 yield:

Corollary 2.7. *For any C^2-differentiable convex surface S with Gaussian curvature $K \geqslant k > 0$, and any simple closed geodesic of length l on S, holds $lD < 2A + \frac{2\pi^2}{k}$.*

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