Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Acad. Sci. Paris, Ser. I 343 (2006) 81–86
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Mathematical Analysis

On the Schur–Szegö composition of polynomials ☆

Vladimir Kostov [a], Boris Shapiro [b]

[a] Laboratoire J.-A. Dieudonné, UMR 6621 du CNRS, Université de Nice, parc Valrose, 06108 Nice cedex 2, France
[b] Mathematics Department, Stockholm University, 106 91 Stockholm, Sweden

Received 29 May 2006; accepted 1 June 2006

Available online 30 June 2006

Presented by Jean-Pierre Kahane

Abstract

The Schur–Szegö composition of two polynomials of degree $\leqslant n$ introduces an interesting semigroup structure on polynomial spaces and is one of the basic tools in the analytic theory of polynomials. In the present Note we show how it interacts with the stratification of polynomials according to the multiplicities of their zeros and we present the induced semigroup structure on the set of all ordered partitions of n. ***To cite this article: V. Kostov, B. Shapiro, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Sur la composition de Schur–Szegö de polynômes. La composition de Schur–Szegö de deux polynômes de degré $\leqslant n$ introduit une structure de semi-groupe dans l'espace de polynômes et est un des outils de base dans la théorie analytique de polynômes. Dans cet article nous montrons comment elle intervient dans la stratification de polynômes selon la multiplicité de leurs racines induisant ainsi une structure de semi-groupe sur l'ensemble des partitions ordonnées d'ordre n. ***Pour citer cet article : V. Kostov, B. Shapiro, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

La composition de Schur–Szegö de deux polynômes $P(x)=\sum_{i=0}^{n} C_n^i a_i x^i$ et $Q(x)=\sum_{i=0}^{n} C_n^i b_i x^i$ est définie par $(P*Q)(x)=\sum_{i=0}^{n} C_n^i a_i b_i x^i$, voir [5]. On désigne par Pol_n l'espace linéaire de tous les polynômes de x de degré au plus n. Dans la suite nous utilisons sa base monomiale standard $(x^n, x^{n-1}, \ldots, 1)$. A tout polynôme $P\in Pol_n$ on associe l'opérateur T_P qui est diagonal dans cette base et qui est complètement défini par la condition : $T_P(1+x)^n = P(x)$. Il est clair que pour $P(x)=C_n^0 a_0 + C_n^1 a_1 x + \cdots + C_n^n a_n x^n$ on a $T_P(x^i)=a_i x^i$, $i=0,1,\ldots,n$. Étant donné P comme ci-dessus on appelle la suite $\{a_i\}$ la *suite diagonale* de P. Tous deux tels opérateurs T_P et T_Q commutent et leur produit $T_P T_Q$ correspond à la composition de Schur–Szegö $P*Q$. Le théorème de composition de Schur et Szegö (voir l'original dans [5] et §3.4 de [4] ou §2 de [1]) est formulé comme suit :

☆ Research partially supported by Wenner-Grenn Foundation.
E-mail addresses: kostov@math.unice.fr (V. Kostov), shapiro@math.su.se (B. Shapiro).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.06.007

Théorème 0.1. *Étant donné l'image $\mathcal{K}$ (qui contient toutes les racines de P) du disque unité par une homographie, toute racine de $P * Q$ est le produit d'une racine de Q et de $-\gamma$ où $\gamma \in \mathcal{K}$.*

Un polynôme $P \in Pol_n$ est dit *hyperbolique* si toutes ses racines sont réelles. On désigne par $Hyp_n \subset Pol_n$ l'ensemble de tous les polynômes hyperboliques et par $Hyp_n^+ \subset Hyp_n$ (resp. $Hyp_n^- \subset Hyp_n$) l'ensemble des polynômes hyperboliques dont toutes les racines sont > 0 (resp. < 0). On désigne par $H_{u,v,w} \subset Hyp_n$ (où $u, v, w \in \mathbf{N} \cup 0$, $u + v + w = n$) l'ensemble des polynômes hyperboliques ayant u racines < 0, w racines > 0 et une racine 0 de multiplicité v.

Proposition 0.2 *(Théorème 5.5.5 et Corollaire 5.5.10 de [4]). Si $P, Q \in Hyp_n$ et si $Q \in Hyp_n^+$ ou $Q \in Hyp_n^-$, alors $P * Q \in Hyp_n$. De plus, toutes les racines de $P * Q$ appartiennent à $[-M, -m]$ où M et m sont le produit maximal et minimal d'une racine de P et d'une racine de Q.*

Une suite diagonale (ou un opérateur $T : Pol_n \to Pol_n$ qui est diagonal dans la base standard) est dite une *suite finie de multiplicateurs de longueur $n + 1$ ($SFM(n + 1)$)*, voir [3], si elle envoie tout polynôme hyperbolique en polynôme hyperbolique. L'ensemble $\mathcal{M}_n$ de toutes les $SFM(n + 1)$ est un semigroupe. La caractérisation suivante des $SFM(n + 1)$ est donnée dans [2], Theorem 3.7 et dans [1], Theorem 3.1.

Théorème 0.3. *Pour $T = \mathrm{diag}(\gamma_n, \ldots, \gamma_0) \in End(Pol_n^{\mathbb{R}})$ les deux conditions suivantes sont équivalentes* :
(i) *T est une $SFM(n + 1)$* ;
(ii) *Toutes les racines non-nulles du polynôme $P_T(x) = \sum_{j=0}^{n} C_n^j \gamma_j x^j$ sont de même signe.*

Dans cette Note on étudie la relation entre les multiplicités des racines de P, Q et $P * Q$.

Proposition 0.4. *Étant donné deux polynômes (complexes) P et Q de degré n tels que x_P, x_Q sont des racines respectivement de P, Q de multiplicité m_P, m_Q où $\mu^* := m_P + m_Q - n \geqslant 0$, on a que $-x_P x_Q$ est racine de $P * Q$ de multiplicité μ^*. (Si $\mu^* = 0$, alors, $-x_P x_Q$ n'est pas racine de $P * Q$.)*

Remarque 1. Si $m_P > 0$, $m_Q > 0$ et $\mu^* < 0$, alors $-x_P x_Q$ peut être ou pas être racine de $P * Q$. Exemple : $((x - 1)(x - 2)(x - 3)) * ((x - 1)(x - 4)(x - d))$ admet -1 comme racine si et seulement si $d = 17/23$.

Proposition 0.5. *Pour tout $P \in H_{u,v,w}$ et pour tout $Q \in Hyp_n^-$ on a $P * Q \in H_{u,v,w}$. En particulier, Hyp_n^- est un semigroupe p.r. à la composition de Schur–Szegö.*

Les racines de P, Q et $P * Q$ de Proposition 0.4 (c. à d. celles de la forme $-x_P x_Q$ la somme des multiplicités de x_P et de x_Q étant $> n$) sont dites *A-racines*, les autres racines de P, Q et $P * Q$ sont dites B-racines. Avec l'exception de 0 – si 0 est racine de P, on le considère comme A-racine de P et de $P * Q$. On associe a tout $P \in Hyp_n$ la partition ordonnée de n dite son *vecteur multiplicité VM_P* obtenu comme l'ensemble ordonné des multiplicités des racines de P dans l'ordre de croissance. Pour α racine de $P \in Hyp_n$ on désigne par $[\alpha]_-$ (resp. $[\alpha]_+$) le nombre total de racines de P à gauche (resp. à droite) p.r. à α et par $\mathrm{sign}(\alpha)$ le signe de α.

Théorème 0.6. *Pour tout $P \in Hyp_n$ et $Q \in Hyp_n^-$ le vecteur multiplicité VM_{P*Q} est défini de façon unique par Proposition* 0.5 *et par les conditions suivantes* :
(i) *Pour toute A-racine $\alpha \neq 0$ de P et toute A-racine β de Q on a $[-\alpha\beta]_- = [\alpha]_- + [\beta]_{\mathrm{sign}(\alpha)}$.*
(ii) *Toutes les B-racines de $P * Q$ sont simples.*

Corollaire 0.7. *La composition de Schur–Szegö restreinte à Hyp_n^- induit une structure de semigroupe dans l'ensemble des partitions ordonnées n. Exemples* : $(2, 14, 1) * (5, 6, 6) = (1, 1, 2, 1, 1, 1, 3, 1, 1, 1, 3, 1)$, $(1, 14, 2) * (5, 6, 4, 2) = (1, 2, 1, 1, 1, 3, 1, 1, 1, 1, 1, 1, 1, 1)$.

1. Introduction

The Schur–Szegö composition of two polynomials

$$P(x) = \sum_{i=0}^{n} C_n^i a_i x^i \quad \text{and} \quad Q(x) = \sum_{i=0}^{n} C_n^i b_i x^i$$

is given by $(P * Q)(x) = \sum_{i=0}^{n} C_n^i a_i b_i x^i$, see e.g. [5]. Let Pol_n denote the linear space of all polynomials in x of degree at most n. In what follows we always use its standard monomial basis $\mathcal{B} := (x^n, x^{n-1}, \dots, 1)$. To any polynomial $P \in Pol_n$ one can associate the operator T_P which acts diagonally in $\mathcal{B}$ and is uniquely determined by the condition: $T_P(1+x)^n = P(x)$. Obviously, for $P(x) = C_n^0 a_0 + C_n^1 a_1 x + \dots + C_n^n a_n x^n$ one has $T_P(x^i) = a_i x^i$, $i = 0, 1, \dots, n$. Given P as above we refer to the sequence $\{a_i\}$ as to the *diagonal sequence* of P. Any two such operators T_P and T_Q commute and their product $T_P T_Q$ corresponds in the above sense exactly to the Schur–Szegö composition $P * Q$. The famous composition theorem of Schur and Szegö (see original [5] and e.g. §3.4 of [4] or §2 of [1]) reads:

Theorem 1.1. *Given any linear-fractional image $\mathcal{K}$ of the unit disk containing all the roots of P one has that any root of $P * Q$ is the product of some root of Q by $-\gamma$ where $\gamma \in \mathcal{K}$.*

Geometric consequences of Theorem 1.1, in particular, Proposition 1.2, can be found in § 5.5 of [4]. A polynomial $P \in Pol_n$ is called *hyperbolic* if all its roots are real. Denote by $Hyp_n \subset Pol_n$ the set of all hyperbolic polynomials and by $Hyp_n^+ \subset Hyp_n$ (resp. $Hyp_n^- \subset Hyp_n$) the set of all hyperbolic polynomials with all positive (resp. all negative) roots. Denote by $H_{u,v,w} \subset Hyp_n$ (where $u, v, w \in \mathbf{N} \cup 0$, $u + v + w = n$) the set of all hyperbolic polynomials with u negative and w positive roots and a v-fold zero root.

Proposition 1.2 *(Theorem 5.5.5 and Corollary 5.5.10 of [4]). If $P, Q \in Hyp_n$ and if $Q \in Hyp_n^+$ or $Q \in Hyp_n^-$, then $P * Q \in Hyp_n$. Moreover, all roots of $P * Q$ lie in $[-M, -m]$ where M is the maximal and m is the minimal pairwise product of roots of P and Q.*

A diagonal sequence, (or an operator $T : Pol_n \to Pol_n$ acting diagonally in $\mathcal{B}$) is called a *finite multiplier sequence of length $n+1$* (*FMS*$(n+1)$), see [3], if it sends Hyp_n into Hyp_n. The set $\mathcal{M}_n$ of all *FMS*$(n+1)$ is a semigroup. For the following characterization of *FMS*$(n+1)$ see [2], Theorem 3.7 or [1], Theorem 3.1.

Theorem 1.3. *For $T = \mathrm{diag}(\gamma_n, \dots, \gamma_0) \in End(Pol_n^{\mathbb{R}})$ the following two conditions are equivalent*:
(i) *T is an FMS$(n+1)$*;
(ii) *All different from* 0 *roots of the polynomial $P_T(x) = \sum_{j=0}^{n} C_n^j \gamma_j x^j$ are of the same sign.*

We get by Theorem 1.3 a linear diffeomorphism of $\mathcal{M}_n$ and $\overline{Hyp}_n^+ \cup \overline{Hyp}_n^-$ where $\overline{X}$ means the closure of X. In this Note we study the relation between the root multiplicities of P, Q and $P * Q$.

Proposition 1.4. *Given two* (*complex*) *polynomials P and Q of degree n such that x_P, x_Q are roots respectively of P, Q of multiplicity m_P, m_Q with $\mu^* := m_P + m_Q - n \geqslant 0$, one has that $-x_P x_Q$ is a root of $P * Q$ of multiplicity μ^*.* (*If $\mu^* = 0$, then $-x_P x_Q$ is not a root of $P * Q$.*)

Remark 1. If $m_P > 0$, $m_Q > 0$ and $\mu^* < 0$, then $-x_P x_Q$ might or might not be a root of $P * Q$. Example: $((x-1)(x-2)(x-3)) * ((x-1)(x-4)(x-d))$ has -1 as a root if and only if $d = 17/23$.

Proposition 1.5. *For any $P \in H_{u,v,w}$ and any $Q \in Hyp_n^-$ one has $P * Q \in H_{u,v,w}$. In particular, Hyp_n^- is a semigroup w.r.t. the Schur–Szegö composition.*

The roots of P, Q and $P * Q$ involved in Proposition 1.4 (i.e. those of the form $-x_P x_Q$, the sum of the multiplicities of x_P and x_Q being $> n$) are called *A-roots*, the remaining roots of P, Q, $P * Q$ are called B-roots. With one exception – if 0 is a root of P, then it is considered as A-root of $P * Q$. Associate to $P \in Hyp_n$ its *multiplicity vector MV_P* (the ordered partition of n defined by the multiplicities of the roots of P in the increasing order). For a root α of $P \in Hyp_n$ denote by $[\alpha]_-$ (resp. $[\alpha]_+$) the total number of roots of P to the left (resp. to the right) of α and by $\mathrm{sign}(\alpha)$ the sign of α.

Theorem 1.6. *For any $P \in Hyp_n$ and $Q \in Hyp_n^-$ the multiplicity vector MV_{P*Q} is uniquely determined by Proposition* 1.5 *and the following conditions*:

(i) *For any A-root* $\alpha \neq 0$ *of* P *and any A-root* β *of* Q *one has* $[-\alpha\beta]_- = [\alpha]_- + [\beta]_{\mathrm{sign}(\alpha)}$.
(ii) *Every B-root of* $P * Q$ *is simple.*

Corollary 1.7. *The Schur–Szegö composition restricted to* Hyp_n^- *induces a semigroup structure on the set of all ordered partitions of n. Examples*: $(2, 14, 1) * (5, 6, 6) = (1, 1, 2, 1, 1, 1, 3, 1, 1, 1, 3, 1)$, $(1, 14, 2) * (5, 6, 4, 2) = (1, 2, 1, 1, 1, 3, 1, 1, 1, 1, 1, 1, 1, 1)$.

2. Proofs

2.1. Proof of Proposition 1.4

Let x_P and x_Q be the required roots of P and Q. It suffices to consider the case $x_P = x_Q = -x_P x_Q = -1$. Indeed, $x_P, x_Q, -x_P x_Q$ are roots of $P(x)$, $Q(x)$, $(P * Q)(x)$ of multiplicities m_P, m_Q, μ^* if and only if 1, 1, -1 are roots of $P(x_P x)$, $Q(x_Q x)$, $(P * Q)(x_P x_Q x)$ of the same multiplicities. Set $G(x) = x^n P(1/x)$. Hence, 1 is an m_P-fold root of G. One has

$$G^{(\nu)}(1) = \frac{n!}{(n-\nu)!}\sum_{j=0}^{n-\nu} C_{n-\nu}^j a_j, \qquad Q^{(\nu)}(1) = \frac{n!}{(n-\nu)!}\sum_{j=0}^{n-\nu} C_{n-\nu}^j b_{j+\nu}.$$

Set

$$K_s := \frac{s!}{n!} G^{(n-s)}(1) = \sum_{q=0}^{s} C_s^q a_q, \qquad L_r := \frac{r!}{n!} Q^{(n-r)}(1) = \sum_{q=0}^{r} C_r^q b_{q+n-r}.$$

Hence, $K_n = K_{n-1} = \cdots = K_{n-m_P+1} = 0 = L_n = L_{n-1} = \cdots = L_{n-m_Q+1}$, $K_{n-m_P} \neq 0 \neq L_{n-m_Q}$. One has

$$\sum_{j=0}^{n} (-1)^j C_n^j K_j L_{n-j} = \sum_{j=0}^{n} (-1)^j C_n^j a_j b_j = (P * Q)(-1). \qquad (*)$$

(Indeed, to prove the equality between two bilinear forms in a_i, b_k, it suffices to set $a_{i_0} = 1$, $a_i = 0$ for $i \neq i_0$, $i_0 = 0, 1, \ldots, n$. The middle part of $(*)$ then equals $(-1)^{i_0} C_n^{i_0} b_{i_0}$, one has $K_j = C_j^{i_0}$ for $j \geqslant i_0$, $K_j = 0$ for $j < i_0$, the left side equals $\sum_{j=i_0}^{n} (-1)^j C_n^j C_j^{i_0} \sum_{\nu=0}^{n-j} C_{n-j}^{\nu} b_{\nu+j}$ $(**)$ and one checks directly that the coefficient before b_l in $(**)$ equals $(-1)^{i_0} C_n^{i_0}$ if $l = i_0$ and 0 if $l \neq i_0$.) Hence, if $\mu^* > 0$, then -1 is a root of $P * Q$ – each product in the left side of $(*)$ contains a zero factor. When $\mu^* = 0$, then all but one products contain such a factor, so $(P * Q)(-1) \neq 0$. To prove that -1 is a root of $P * Q$ of multiplicity μ^* one has to show for $\lambda < \mu^*$ (by analogy with $(*)$) that

$$\sum_{j=0}^{n-\lambda} (-1)^j C_{n-\lambda}^j K_j^\lambda L_{n-\lambda-j} = \sum_{j=0}^{n-\lambda} (-1)^j C_{n-\lambda}^j a_{j+\lambda} b_{j+\lambda} = \frac{(n-\lambda)!}{n!} (P * Q)^{(\nu)}(-1)$$

where $K_j^\lambda = \sum_{q=0}^{j} C_j^q a_{q+\lambda} = \frac{j!}{n!} (x^{n-\lambda} P^{(\lambda)}(1/x))^{(n-\lambda-j)}|_{x=1}$.

2.2. Proof of Proposition 1.5

We prove it in the case $v = 0$, i.e for any $P \in Hyp_n$, $P(0) \neq 0$ and any $Q \in Hyp_n^-$. The general case follows by continuity. The statement is trivially true for any $P \in Hyp_n$ and $Q(x) = (1 + x)^n$ since $P * Q = P$. Let $Q \in Hyp_n^-$. Connect $Q(x)$ to $(1 + x)^n$ by some path $Q^t(x)$ within Hyp_n^-. (This is possible since Hyp_n^- is contractible.) Notice that if $P(0) \neq 0$, then $(P * Q^t)(0) \neq 0$ for the whole family since the constant term of $P * Q$ is the product of the ones of P and Q. Therefore the number of positive and negative roots of $P * Q$ is the same as for $P * (1 + x)^n$.

2.3. Proof of Theorem 1.6(i)

Instead of $(P * Q)(x)$ we consider $Z(x) := (P * Q)(-x)$ (to have the same ordering of the roots on the line in all three polynomials). Suppose that $P \in H_{s,l,n-s-l}$, i.e. $P = (\prod_{j=1}^{s}(x + a_j))x^l(\prod_{j=s+l+1}^{n}(x - a_j))$, $a_j > 0$. Fix a_j for

$j = 1, \ldots, s$ and deform them continuously into 0 for $j = s + l + 1, \ldots, n$. The MV of the negative root sets of Z does not change. Therefore to find this MV it suffices to find it for P replaced by $P_1 := (\prod_{j=1}^{s}(x + a_j))x^{n-s}$. In the same way, to find the MV of the positive root sets of Z it suffices to find it for P replaced by $P_2 := x^{s+l}(\prod_{j=s+l+1}^{n}(x - a_j))$. When $P(x)$ is changed to $P(-x)$, then $Z(x)$ changes to $Z(-x)$ (this explains the presence of $\mathrm{sign}(\alpha)$ in (i)), and the description of the MV of the root sets of Z can be done by considering only polynomials of the form P_2.

Consider the case when $P, Q \in Hyp_n^+$ (hence, P and Q play the same role). The other three cases $P \in Hyp_n^{\pm}$, $Q \in Hyp_n^{\pm}$ can be treated by analogy using $P(-x) * Q(x) = P(x) * Q(-x) = (P * Q)(-x)$. If $P = (x - a)^n$, then $Z(x) = Q(ax)$, so assume that each polynomial P, Q has two distinct roots, $0 < a_1 < a_2$ and $0 < b_1 < b_2$, of multiplicities m_1, m_2 and n_1, n_2. If n is even and $m_1 = n_1 = n/2$, then Z has no A-roots. Recall that by Proposition 1.4 if $a_i b_j$ is an A-root, then its multiplicity is $m_i + n_j - n$.

Assume that (one of) the biggest of the four multiplicities m_1, m_2, n_1, n_2 is among the last two. Suppose first that this is n_1. If $n_1 + m_1 > n$, $n_1 + m_2 > n$, then the root set $R(Z)$ of Z looks like this: (a_1b_1, V, a_2b_1, Y), see Propositions 1.4 and 1.2. Set $\sharp(V) = v$, $\sharp(Y) = y$. When writing $b_1 \to 0$ or $b_1 \to b_2$ we mean that the roots a_1, a_2, b_2 are fixed. When $b_1 \to 0$, then in the limit Z has n_2 non-zero roots which are all from Y, hence, $y \geqslant n_2$. When $b_2 \to b_1$, then in the limit $Z(x) = P(b_1x)$ has two roots, of multiplicities m_1 and m_2. Hence, $(n_1 + m_1 - n) + v \geqslant m_1$, i.e. $v \geqslant n_2$. But $v + y = 2n_2$, hence, $v = y = n_2$.

If $n_1 + m_1 > n \geqslant n_1 + m_2$, then $R(Z) = (a_1b_1, V)$, $v = n_2 + m_2$. If $n_1 + m_1 \leqslant n < n_1 + m_2$, then $R(Z) = (U, a_2b_1, V)$, $u + v = n_2 + m_1$. When $b_1 \to 0$, then $v \geqslant n_2$ because all n_2 non-zero (in the limit) roots are in V. When $b_2 \to b_1$, then $(n_1 + m_2 - n) + v \leqslant m_2$, i.e. $v \leqslant n_2$. Hence, $v = n_2$, $u = m_1$.

Let $n_2 = \max(m_1, m_2, n_1, n_2)$. If $n_2 + m_1 > n$, $n_2 + m_2 > n$, then $R(Z) = (U, a_1b_2, V, a_2b_2)$. When $b_1 \to 0$, this yields $u \geqslant n_1$, and $b_1 \to b_2$ yields $v \geqslant n_1$. As $u + v = 2n_1$, one has $u = v = n_1$. If $n_2 + m_1 > n \geqslant n_2 + m_2$, then $R(Z) = (U, a_1b_2, V)$. When $b_1 \to 0$, this yields $u \geqslant n_1$, and $a_1 \to 0$ implies $v \geqslant m_2$. As $u + v = m_2 + n_1$, one has $u = n_1$, $v = m_2$. If $n_2 + m_1 \leqslant n < n_2 + m_2$, then $R(Z) = (U, a_2b_2)$, $u = m_1 + n_1$. This proves (i) of Theorem 1.6 for $P, Q \in Hyp_n^+$ having each $\leqslant 2$ distinct roots.

Further we assume that P has a single A-root a, of multiplicity m. To prove the theorem by induction on the number of distinct positive roots in P and Q it suffices to consider the result on the MV of the root sets of Z when a multiple root of P or Q splits into two. If this is a B-root, such a splitting deforms continuously the B-roots in Z, its A-roots and their multiplicities don't change, and the theorem holds.

If an A-root splits into two B-roots, then it is a root of Q. Suppose that this is b_{j_d}, and that one has $b_{j_{d-1}} < b_{j_d} < b_{j_{d+1}}$, b_{j_ν} being A-roots. Denote the multiplicities of these three roots by h_1, h_2, h_3, and by t_1, t_2 the sums of the multiplicities of the B-roots of Q from $(b_{j_{d-1}}, b_{j_d})$ and $(b_{j_d}, b_{j_{d+1}})$. Before the splitting of b_{j_d} the polynomial Z had three A-roots stemming from $b_{j_{d-1}} < b_{j_d} < b_{j_{d+1}}$, namely, $ab_{j_{d-1}} < ab_{j_d} < ab_{j_{d+1}}$, of multiplicities $m + h_i - n$, $i = 1, 2, 3$, with sums of the multiplicities of the B-roots of Z from the two intervals between them equal to $t_1 + n - m$, $t_2 + n - m$. After the splitting there remain only the A-roots $ab_{j_{d-1}} < ab_{j_{d+1}}$, the A-root ab_{j_d} splits into B-roots of total multiplicity $m + h_2 - n$. In Q there remain the A-roots $b_{j_{d-1}} < b_{j_{d+1}}$ with total multiplicity of the B-roots between them equal to $t_1 + t_2 + h_2$. Thus the sum of the multiplicities of the B-roots of Z from $(ab_{j_{d-1}}, ab_{j_{d+1}})$ after the splitting equals $t_1 + t_2 - 2m + 2n + m + h_2 - n = t_1 + t_2 + h_2 + n - m$. Hence, (i) of Theorem 1.6 holds after the splitting. If the A-root b_{j_d} is first or last, i.e. $d = 1$ or $d = r$, then the proof is similar.

Suppose that an A-root (say, c of P, of multiplicity μ) is splitting into an A-root to the left and a B-root to the right, of multiplicities ξ and η. Then in Z there is a splitting of an A-root cf (f is an A-root of Q) of multiplicity $\mu + \nu - n$ into an A-root of multiplicity $\xi + \nu - n$ and one or several B-roots of total multiplicity η. Suppose that at least one of these B-roots goes to the left. Shift to the left (after the splitting) all roots of P simultaneously while keeping the ones of Q fixed. When one has $c = 0$, then the number of positive roots (counted with the multiplicities) will be greater for P than for Z (this follows from $[cf]_+ = [c]_+ + [f]_+$ before the splitting). This is a contradiction with Proposition 1.5. Hence, all new B-roots of Z go to the right after the splitting and one checks directly that (i) of Theorem 1.6 holds after the splitting. If the B-root of P goes to the left, or if c is a root of Q, then the reasoning is similar.

If an A-root c splits into two A-roots c^1 (left) and c^2 (right) (hence, c is a root of Q), then the above reasoning shows that in Z an A-root cf splits into two A-roots $c^1 f$ (left, f is an A-root of P) and $c^2 f$ (right) and one or several B-roots between them. Indeed, one shows as above that all roots different from $c^1 f$ (resp. $c^2 f$) and stemming from cf must go right (resp. left). Hence, the B-roots of Z resulting from the splitting are between $c^1 f$ and $c^2 f$. Denote by n^0, n^1, n^2 and m^0 the multiplicities of c, c^1, c^2 and f ($n^0 = n^1 + n^2$). Hence, the multiplicities of $c^1 f$, $c^2 f$ and the total multiplicity of the B-roots of Z between them equal n^1+m^0-n, n^2+m^0-n and $n-m^0$, i.e. (i) of Theorem 1.6 holds after the splitting.

2.4. Proof of Theorem 1.6(ii)

We show that non-simplicity of a B-root contradicts $P*Q\in Hyp_n$ for any $P\in Hyp_n$, $Q\in Hyp_n^{\pm}$, see Proposition 1.2. We first settle the basic case when either P or Q has only simple zeros and then use a procedure which either decreases the multiplicity of some root of P or leads to $P*Q\notin Hyp_n$. The multiplicity of 0 as a root of P must decrease up to 0, not to 1.

(1) Basic case. Suppose that $b\neq 0$ is a B-root of $P*Q$ of multiplicity $\mu\geqslant 2$. If P has distinct real non-zero roots, then such are all polynomials from a small neighbourhood Δ of P in $Pol_n^{\mathbb{R}}$. If $\mu=2$, then adjusting the constant term which is non-zero by assumption one can easily choose $T\in\Delta$ such that $T*Q$ have a complex conjugate pair of zeros close to b – a contradiction. If $\mu>2$, then one can choose T such that $(T*Q)'$ has a multiple root at b and $(T*Q)(b)\neq 0$. Hence, $T*Q\notin Hyp_n$.

(2) General case. Assume that $P=(x-c)^l P_1(x)$, $l\geqslant 2$, $P_1(c)\neq 0$. Set $P_c(x):=P(x)/(x-c)$. Consider the family of hyperbolic polynomials $P^\delta=P+\delta P_c$, $\delta\in\mathbb{R}$ (†). In this family the l-tuple root c splits into the $(l-1)$-tuple root c and an extra root $c-\delta$ which is simple unless it coincides with some other root of P. Set $U_c:=P_c*Q$. Further considerations split into subcases (2.i)–(2.iii) below:

(2.i) If $U_c(b)\neq 0$, then we find a value of δ such that $P^\delta*Q\notin Hyp_n$ – a contradiction. Indeed, if μ is even, then choosing $\operatorname{sign}(\delta)$ one obtains that $P^\delta*Q$ has no real roots close to b. If μ is odd, choose its sign so that the total multiplicity of the roots of $P*Q$ close to b is $<\mu$ (when $U_c'(b)\neq 0$, then $P*Q$ can be made monotonous close to b; when $U_c'(b)=0\neq U_c''(b)$, then choose $\operatorname{sign}(\delta)$ so that there is a local minimum (maximum) of $P*Q$ close to b where $P*Q$ is positive (negative); if $U_c'(b)=U_c''(b)=0$, then for $\delta\neq 0$, b is a degenerate critical point and a non-root of $P*Q$, hence, $P*Q\notin Hyp_n$).

(2.ii) If $U_c(b)=U_c'(b)=0$, then $P^\delta*Q$ still has a multiple B-root at b and lower multiplicity of c.

(2.iii) Suppose that $U_c(b)=0\neq U_c'(b)$. Assume first that this happens for at least two distinct roots c, d of P of which c is multiple or $c=0$. Set $P_{cd}:=P/((x-c)(x-d))$. Consider the 2-parameter family of hyperbolic polynomials $P^{\delta,\varepsilon}(x)=P(x)+\delta P_c(x)+\varepsilon P_d(x)+\delta\varepsilon P_{cd}(x)$ (‡). In this family the root c (and also d when multiple) splits as in (†). Observe that $P_{cd}=(P_c-P_d)/(c-d)$ (+). Hence, $(P_{cd}*Q)(b)=0$. Set $\delta=-\varepsilon(P_d*Q)'(b)/((P_c+\varepsilon P_{cd})*Q)'(b)$. For $\varepsilon\neq 0$ small enough one has $((P_c+\varepsilon P_{cd})*Q)'(b)\neq 0$. With this choice of δ the polynomial $P^{\delta,\varepsilon}*Q$ still has a multiple root at b and lower multiplicity of c.

(2.iv) To finish the argument notice that the only case to consider when one cannot perform splittings of roots of P is when P has a single multiple or zero root c (of multiplicity ν) with $U_c(b)=0\neq U_c'(b)$, and the remaining non-zero roots d_i of P are all simple with $(P_{d_i}*Q)(b)=(P_{d_i}*Q)'(b)=0$. The same must be true for Q; denote by g the root of Q of multiplicity $\lambda>1$. But then a suitable linear combination of P and P_{d_i} equals $(x-c)^n$ (see (+) etc.). Hence, $(x-c)^n*Q=Q(cx)$ has a multiple root at b, i.e. $b=cg$. As b must be a B-root (and not an A-root) of $P*Q$, one must have $\nu+\lambda\leqslant n$. If $\nu+\lambda=n$, then b is a non-root of $P*Q$ by Proposition 1.4 – a contradiction. If $\nu+\lambda<n$, then a suitable linear combination of P and P_{d_i} equals $Y:=(x-h)^\lambda(x-c)^{n-\lambda}$ for which one has that b is a multiple root of $Y*Q$ – a contradiction with Proposition 1.4 again.

References

[1] T. Craven, G. Csordas, Composition theorems, multiplier sequences and complex zero decreasing sequences, in: Value Distribution Theory and Its Related Topics, G. Barsegian, I. Laine, C.C. Yang (Eds.), Kluwer Academic Press, in press.

[2] T. Craven, G. Csordas, Multiplier sequences for fields, Illinois J. Math. 21 (4) (1977) 801–817.

[3] G. Pólya, J. Schur, Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen, J. Reine Angew. Mat. 144 (1914) 89–113.

[4] Q.I. Rahman, G. Schmeisser, Analytic Theory of Polynomials, London Math. Soc. Monogr. (N.S.), vol. 26, Oxford Univ. Press, New York, NY, 2002.

[5] G. Szegö, Bemerkungen zu einem Satz von J.H. Grace über die Wurzeln algebraischer Gleichungen, Math. Z. (2) 13 (1922) 28–55.